

2.9. Integration by partial fraction method

Using when partial fraction only formula $\frac{f(x)}{g(x)}$.

When the degree of the numerator is less than the degree of the denominator. There are four states.

First state.

If the factors of $g(x)$ are linear and not repeated.

$$\int \frac{dx}{(ax+b)(ax-b)} = \int \frac{A}{(ax+b)} dx + \int \frac{B}{(ax-b)} dx$$

Example 1: Find $\int \frac{5X-3}{(X+1)(X-3)} dx$.

Solution /

$$\int \frac{5X-3}{(X+1)(X-3)} dx = \int \frac{A}{(X+1)} dx + \int \frac{B}{(X-3)} dx$$

$$\frac{5X-3}{(X+1)(X-3)} = \frac{A(X-3) + B(X+1)}{(X+1)(X-3)}$$

$$5X-3 = AX-3A+BX+B$$

$$A+B=5 \dots \dots \dots (1)$$

$$B-3A=-3 \dots \dots \dots (2)$$

$$5-A-3A=-3 \rightarrow A=2 \rightarrow B=3$$

$$\int \frac{5X-3}{(X+1)(X-3)} dx = \int \frac{2}{(X+1)} dx + \int \frac{3}{(X-3)} dx$$

$$= 2 \ln|x + 1| + 3 \ln|x - 3| + c$$

Exercise:1. $\int \frac{5x-7}{(x-1)(x-2)} dx$

Second state.

If the factors of $g(x)$ are linear and repeated

$$\int \frac{dx}{(ax + b)^2} = \int \frac{A}{(ax + b)^1} dx + \int \frac{B}{(ax + b)^2} dx$$

Example 2: Find $\int \frac{x^2}{(x-1)(x+1)^2} dx$.

Solution /

$$= \int \frac{x^2}{(x-1)(x+1)^2} dx = \int \frac{A}{(x-1)} dx + \int \frac{B}{(x+1)} dx + \int \frac{C}{(x+1)^2} dx$$

$$\frac{x^2}{(x-1)(x+1)^2} = \frac{A(X+1)^2 + B(X-1)(X+1) + C(X-1)}{(x-1)(x+1)^2}$$

$$X^2 = AX^2 + 2AX + A + BX^2 - B + CX - C$$

$$A+B=1 \dots\dots(1)$$

$$2A+C=0 \dots\dots(2)$$

$$A-B-C=0 \dots\dots(3)$$

From (2)&(3) $A-B+2A=0 \rightarrow 3A - B = 0$ SUB IN ... (1)

$$A = \frac{1}{4} \quad B = \frac{3}{4} \quad C = \frac{-1}{2} \therefore$$

$$= \int \frac{\frac{1}{4}}{(X-1)} dx + \int \frac{\frac{3}{4}}{(X+1)} dx + \int \frac{\frac{-1}{2}}{(X+1)^2} dx$$
$$\frac{1}{4} \ln|x-1| + \frac{3}{4} \ln|x+1| + \frac{1}{2(x+1)} + c$$

Third state.

If the factors of $g(x)$ are not linear and not repeated.

$$\int \frac{dx}{(ax^2 + b)(ax^2 - b)} = \int \frac{Ax + B}{(ax^2 + b)} dx + \int \frac{Cx + D}{(ax^2 - b)} dx$$

Example 3: Find $\int \frac{1}{(x^2+1)(x+1)} dx$

Solution/

$$\int \frac{1}{(x^2 + 1)(x + 1)} dx = \int \frac{Ax + B}{(x^2 + 1)} dx + \int \frac{C}{(x + 1)} dx$$

$$\frac{1}{(x^2 + 1)(x + 1)} = \frac{Ax + B(x + 1) + C(x^2 + 1)}{(x^2 + 1)(x + 1)}$$

$$1 = Ax^2 + Ax + Bx + B + Cx^2 + C$$

$$A+C=0.....(1)$$

$$A+B=0.....(2)$$

$$B+C=1.....(3)$$

From (1) ,(2)&(3)

$$A = \frac{-1}{2} \quad B = \frac{1}{2} \quad C = \frac{1}{2} \therefore$$

$$\begin{aligned} \int \frac{1}{(x^2 + 1)(x + 1)} dx &= \int \frac{\frac{-1}{2}x + \frac{1}{2}}{(x^2 + 1)} dx + \int \frac{\frac{1}{2}}{(x + 1)} dx \\ &= \frac{-1}{2} \int \frac{x}{(x^2 + 1)} dx + \frac{1}{2} \int \frac{dx}{(x^2 + 1)} dx + \frac{1}{2} \int \frac{dx}{(x + 1)} \\ &= \frac{-1}{4} \ln|x^2 + 1| + \frac{1}{2} \tan^{-1} x + \frac{1}{2} \ln|x + 1| + c \end{aligned}$$

Four state.

If the factors of $g(x)$ are not linear and repeated.

$$\int \frac{dx}{(ax^2 + b)^2} = \int \frac{Ax + B}{(ax^2 + b)^1} dx + \int \frac{Cx + D}{(ax^2 + b)^2} dx$$

Example 4: Find $\int \frac{(x^2 + 1)}{(x^2 + 2x + 3)^2} dx$

Solution/

$$\begin{aligned} \int \frac{(x^2 + 1)}{(x^2 + 2x + 3)^2} dx \\ &= \int \frac{Ax + B}{(x^2 + 2x + 3)} dx + \int \frac{Cx + D}{(x^2 + 2x + 3)^2} dx \\ \frac{(x^2 + 1)}{(x^2 + 2x + 3)^2} &= \frac{Ax + B(x^2 + 2x + 3) + Cx + D}{(x^2 + 2x + 3)^2} \end{aligned}$$

$$x^2 + 1 = Ax^3 + 2Ax^2 + 3Ax + Bx^2 + 2Bx + 3B + Cx + D$$

$$A = 0$$

$$2A+B=1 \rightarrow B=1$$

$$3A+2B+C=0 \rightarrow C=-2$$

$$3B+D=1 \rightarrow D = -2$$

$$\begin{aligned} \int \frac{(x^2 + 1)}{(x^2 + 2x + 3)^2} dx \\ = \int \frac{1}{(x^2 + 2x + 3)} dx + \int \frac{-2x - 2}{(x^2 + 2x + 3)^2} dx \end{aligned}$$

$$= \int \frac{1}{(x+1)^2 + (\sqrt{2})^2} dx + \int (-2x - 2)(x^2 + 2x + 3)^{-2} dx$$

$$= \frac{1}{\sqrt{2}} \tan^{-1} \frac{(x+1)}{\sqrt{2}} + \frac{(x^2+2x+3)^{-1}}{1} + c$$