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كلية العلـــــوم
قـــــــــــــم الانــظـــمـــة الـــطـبـيـة الـــذكـــــيـــة

المحاضرة الثالثة

Digital Signal Processing

المادة : DSP
المرحلة : الثالثة
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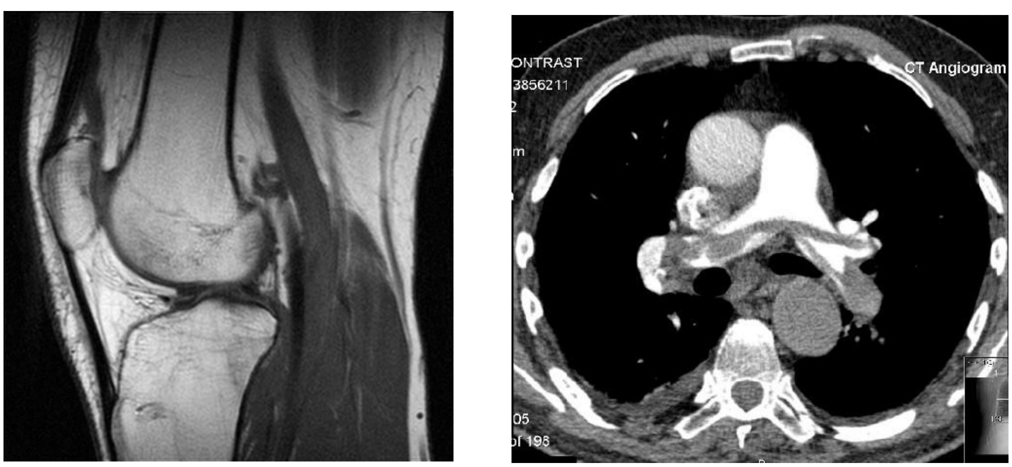
Discrete Fourier Transform (DFT) and Fast Fourier Transform (FFT)

3.1 DFT

In time domain, representation of digital signals describes the signal amplitude versus the sampling time instant or the sample number. However, in some applications, signal frequency content is very useful than as digital signal samples. The algorithm transforming the time domain signal samples to the frequency domain components is known as the discrete Fourier transform, or DFT. The DFT also establishes a relationship between the time domain representation and the frequency domain representation. Therefore, we can apply the DFT to perform frequency analysis of a time domain sequence. In addition, the DFT is widely used in many other areas, including spectral analysis, acoustics, imaging/ video, audio, instrumentation, communications systems and Medical imaging (X-ray computed tomography (CT) and Magnetic resonance imaging (MRI)).



a b

Fig.1: a- Magnetic resonance imaging (MRI), b- X-ray computed tomography (CT)

DFT Formulas

Given a sequence x(n), $0 \le n \le N-1$, its DFT is defined as:

$$x(k)=\sum_{n=0}^{N-1} x(n)e^{-2\pi nkj/N} = \sum_{n=0}^{N-1} x(n)\, W_N^{kn},\ for\ k=0,1,..N-1$$

Where the factor W_N(called the twiddle factor in some textbooks) is defined as

$$W_N = e^{-j\frac{2\pi}{N}} = \cos\left(\frac{2\pi}{N}\right) - j\sin\left(\frac{2\pi}{N}\right)$$

The inverse DFT is given by:

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X(k)\, e^{j\frac{2\pi kn}{N}} = \frac{1}{N}\sum_{n=0}^{N-1} X(k)\, W_N^{-kn},\ for\ n=0,1,..N-1$$

Example (1): Given a sequence **x(n)** for **$0 \le n \le 3$**, where **x(0) = 1, x(1) = 2, x(2) = 3**, and **x(3) = 4**. Evaluate its DFT X(k).

Solution:

Since N=4, $W_4 = e^{-j\pi/2}$, then using:

$$X(k) = \sum_{n=0}^{3} x(n) W_4^{kn} = \sum_{n=0}^{3} x(n) e^{-j\frac{\pi kn}{2}}.$$

Thus, for $k = 0$

$$X(0) = \sum_{n=0}^{3} x(n)e^{-j0} = x(0)e^{-j0} + x(1)e^{-j0} + x(2)e^{-j0} + x(3)e^{-j0}$$
$$= x(0) + x(1) + x(2) + x(3)$$
$$= 1 + 2 + 3 + 4 = 10$$

for $k = 1$

$$X(1) = \sum_{n=0}^{3} x(n)e^{-j\frac{\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{\pi}{2}} + x(2)e^{-j\pi} + x(3)e^{-j\frac{3\pi}{2}}$$
$$= x(0) - jx(1) - x(2) + jx(3)$$
$$= 1 - j2 - 3 + j4 = -2 + j2$$

for $k = 2$

$$X(2) = \sum_{n=0}^{3} x(n)e^{-j\pi n} = x(0)e^{-j0} + x(1)e^{-j\pi} + x(2)e^{-j2\pi} + x(3)e^{-j3\pi}$$
$$= x(0) - x(1) + x(2) - x(3)$$
$$= 1 - 2 + 3 - 4 = -2$$

and for $k = 3$

$$X(3) = \sum_{n=0}^{3} x(n)e^{-j\frac{3\pi n}{2}} = x(0)e^{-j0} + x(1)e^{-j\frac{3\pi}{2}} + x(2)e^{-j3\pi} + x(3)e^{-j\frac{9\pi}{2}}$$
$$= x(0) + jx(1) - x(2) - jx(3)$$
$$= 1 + j2 - 3 - j4 = -2 - j2$$

Example (2): Find the inverse DFT for X(k) in Example 2 to determine the time domain sequence **x(n).**

Solution:

a. Since $N = 4$ and $W_4^{-1} = e^{j\frac{\pi}{2}}$, using Equation (4.10) we achieve a simplified formula,

$$x(n) = \frac{1}{4}\sum_{k=0}^{3} X(k)W_4^{-nk} = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{\pi kn}{2}}.$$

Then for $n = 0$

$$x(0) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j0} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j0} + X(2)e^{j0} + X(3)e^{j0}\right)$$
$$= \frac{1}{4}(10 + (-2 + j2) - 2 + (-2 - j2)) = 1$$

for $n = 1$

$$x(1) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{k\pi}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{\pi}{2}} + X(2)e^{j\pi} + X(3)e^{j\frac{3\pi}{2}}\right)$$
$$= \frac{1}{4}(X(0) + jX(1) - X(2) - jX(3))$$

$$= \frac{1}{4}(10 + j(-2 + j2) - (-2) - j(-2 - j2)) = 2$$

for $n = 2$

$$x(2) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{jk\pi} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\pi} + X(2)e^{j2\pi} + X(3)e^{j3\pi}\right)$$
$$= \frac{1}{4}(X(0) - X(1) + X(2) - X(3))$$
$$= \frac{1}{4}(10 - (-2 + j2) + (-2) - (-2 - j2)) = 3$$

and for $n = 3$

$$x(3) = \frac{1}{4}\sum_{k=0}^{3} X(k)e^{j\frac{k\pi 3}{2}} = \frac{1}{4}\left(X(0)e^{j0} + X(1)e^{j\frac{3\pi}{2}} + X(2)e^{j3\pi} + X(3)e^{j\frac{9\pi}{2}}\right)$$

$$= \frac{1}{4}(X(0) - jX(1) - X(2) + jX(3))$$

$$= \frac{1}{4}(10 - j(-2 + j2) - (-2) + j(-2 - j2)) = 4$$

H.W:

1- Find the DFT X(k) for the sequence x=[1 3 5 2].

3.2 DFT PROPERTIES

In this section, we list some of the properties of the DFT. Because each sequence is assumed to be finite in length, some care must be exercised in manipulating DFTs.

1. **Linearity**
2. **Symmetry**
3. **Circular Shift**
4. **Circular Convolution**
5. **Multiplication**

3.3 FFT

Definition: The discrete FFT is an algorithm that converts a sampled complex-valued function of time into a sampled complex-valued function of frequency. An FFT computes the DFT and produces exactly the same result as evaluating the DFT definition directly; the only difference is that an FFT is much faster. (In the presence of round-off error, many FFT algorithms are also much more accurate than evaluating the DFT definition directly.

There are basically two types of FFT algorithms:

a. **Decimation in Time**
b. **Decimation in Frequency**

- The most straightforward way to derive the FFT is to separate the DFT summation index into odd and even indexed summations, substitute for the summation index, and recognize that there are then DFTs for X_{even} and X_{odd} as shown in Fig(2).

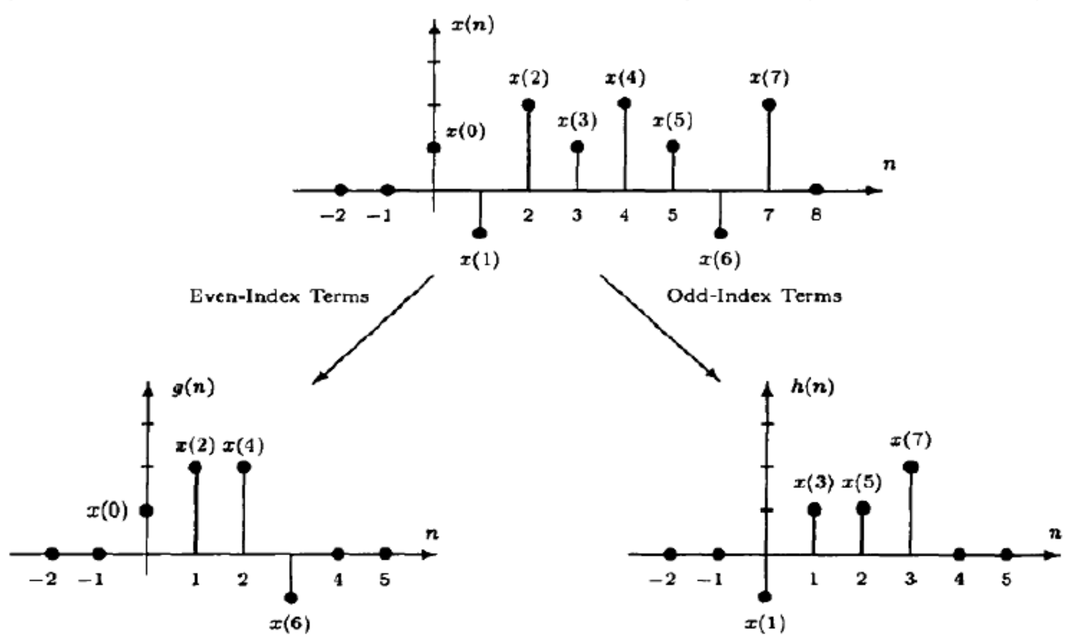


Fig.2

a. Decimation in Time

- **DIT** algorithm is used to calculate the DFT of N-point sequence. The idea is to break the N-point sequence into two sequence, the DFTs of which can be obtained to give the DFT of the original N-point sequence.
- Initially the N-point sequence is divided into N/2 point sequences X_{even} and X_{odd} , The N/2-point DFTs of these two sequences are evaluated and combined to give the N-point DFT. Similarly the N/2-point can be expressed as a combination of N/4-point DFTs.
- If N is a power of 2, the decimation may be continued until there are only two-point DFTs of the form shown in Fig.(3).

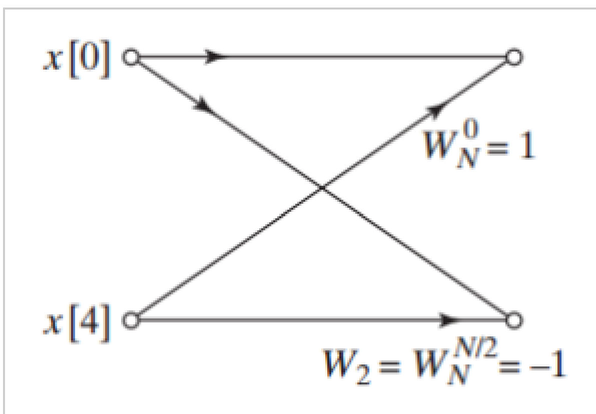


Fig.3 The flow graph used to compute the uppermost two-point DFT

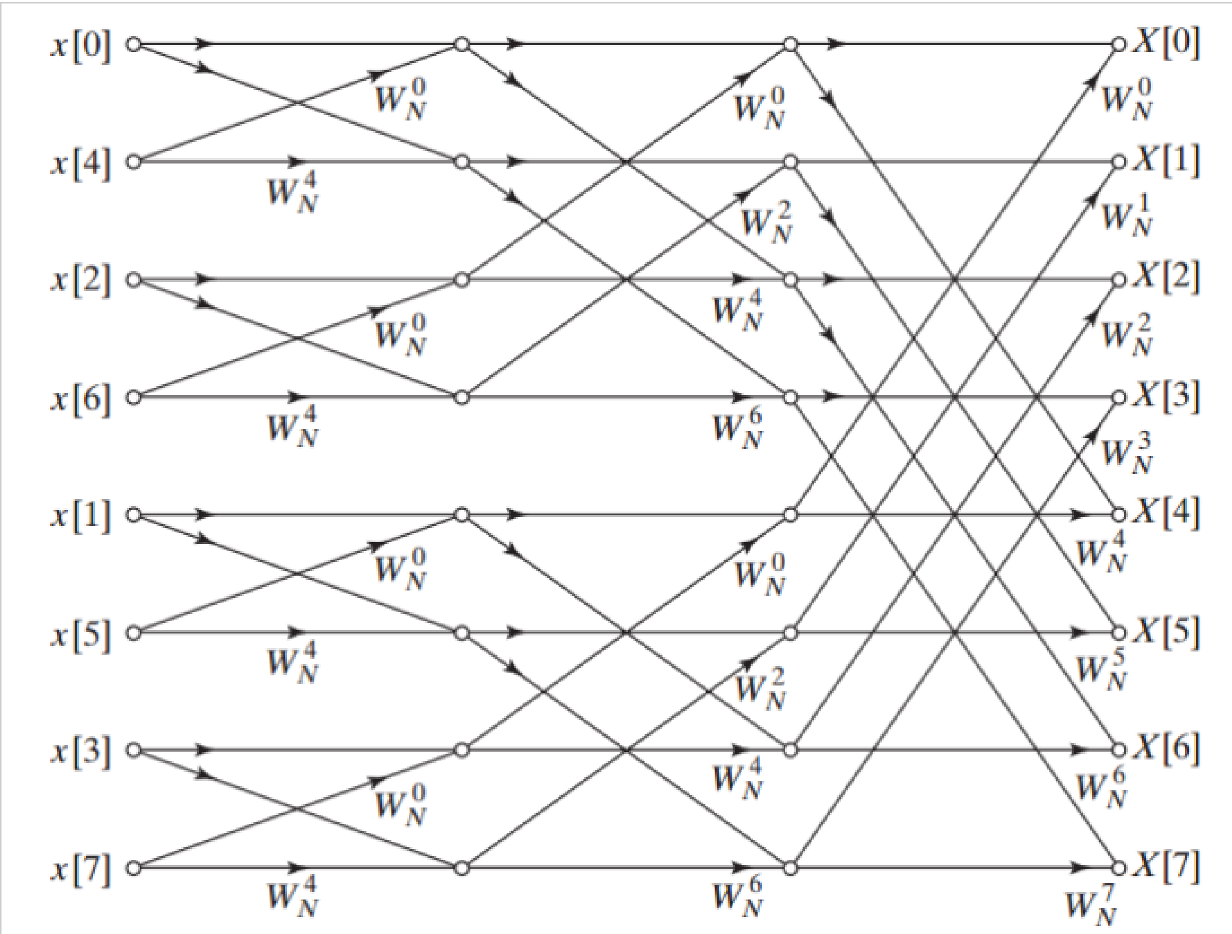


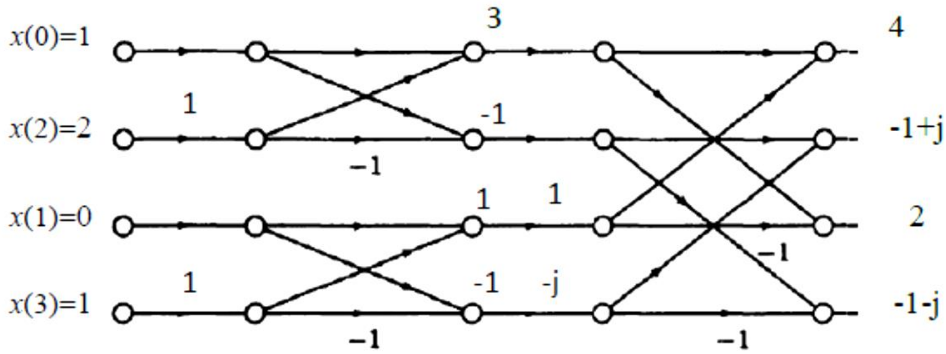
Fig.4 The decimation-in-time based structure to compute an eight-point DFT.

The basic computational unit of the FFT, shown in Fig.(3), is called a *butterfly*. This structure may be simplified by factoring out a term from the lower branch as illustrated in Fig.(3). The factor that remains is complete eight-point radix-2 decimation-in-time FFT is shown in Fig.(4).

Example (3): Find the DFT for x= [1 0 2 1] using the FFT algorithm.

<u>Solution:</u> The scale factor W_4^k (k=0,1,…, N/2-1), Where N=4.

$$W_4^0 = 1, W_4^1 = -j$$

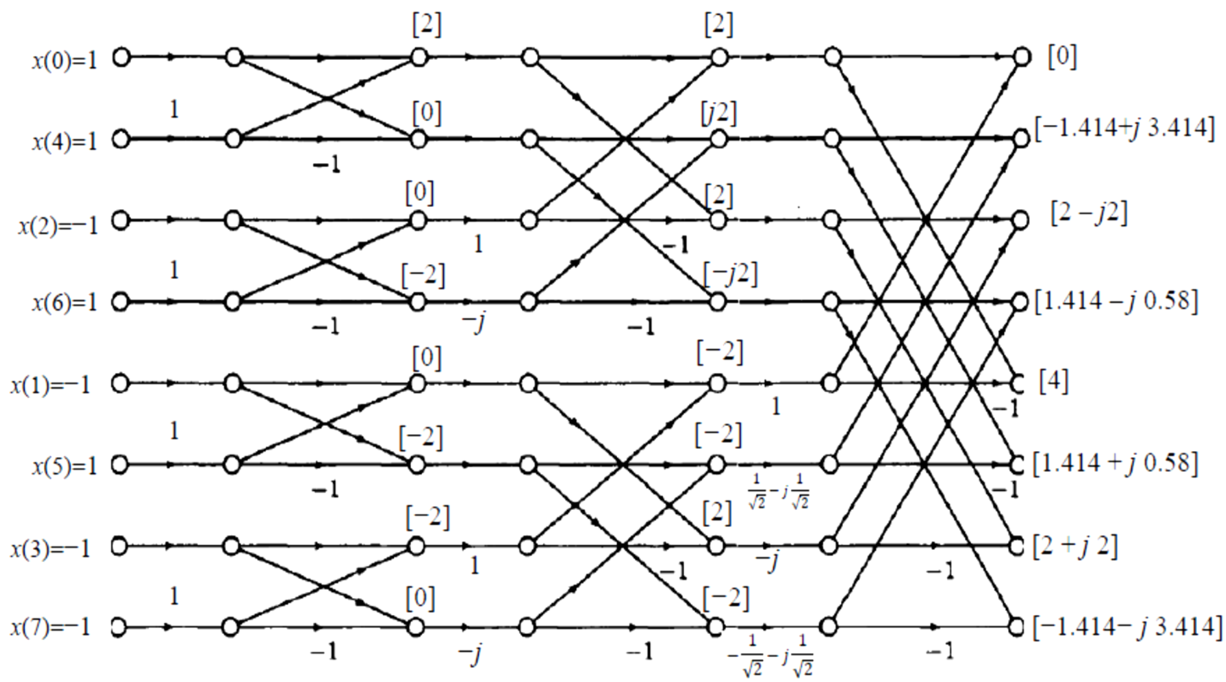


Example (4): Find the DFT for the following sequence x using the FFT algorithm.

$$x = [1, -1, -1, -1, 1, 1, 1, -1]$$

<u>Solution:</u> The scale factor W_8^k (k=0,1,…, N/2-1), Where N=8.

$$W_8^0 = 1,\ W_8^1 = e^{-j2\pi/8} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}, W_8^2 = e^{-j4\pi/8} = -j, \quad W_8^3 = e^{-j2\pi/8} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}},$$



H.W: Find DIT-FFT algorithm:

a) x (n) = [0 , 1 , 2 , 3]

b) x(n)= [1 3 5 2]

b. Decimation in Frequency (FFT)

Another class of FFT algorithms may be derived by decimating the output sequence X(k) into smaller and smaller subsequences. These algorithms are called decimation-in-frequency FFTs and may be derived as follows. Let N be a power of 2, .N= 2^L and consider separately evaluating the even-index and odd-index samples of X(k).

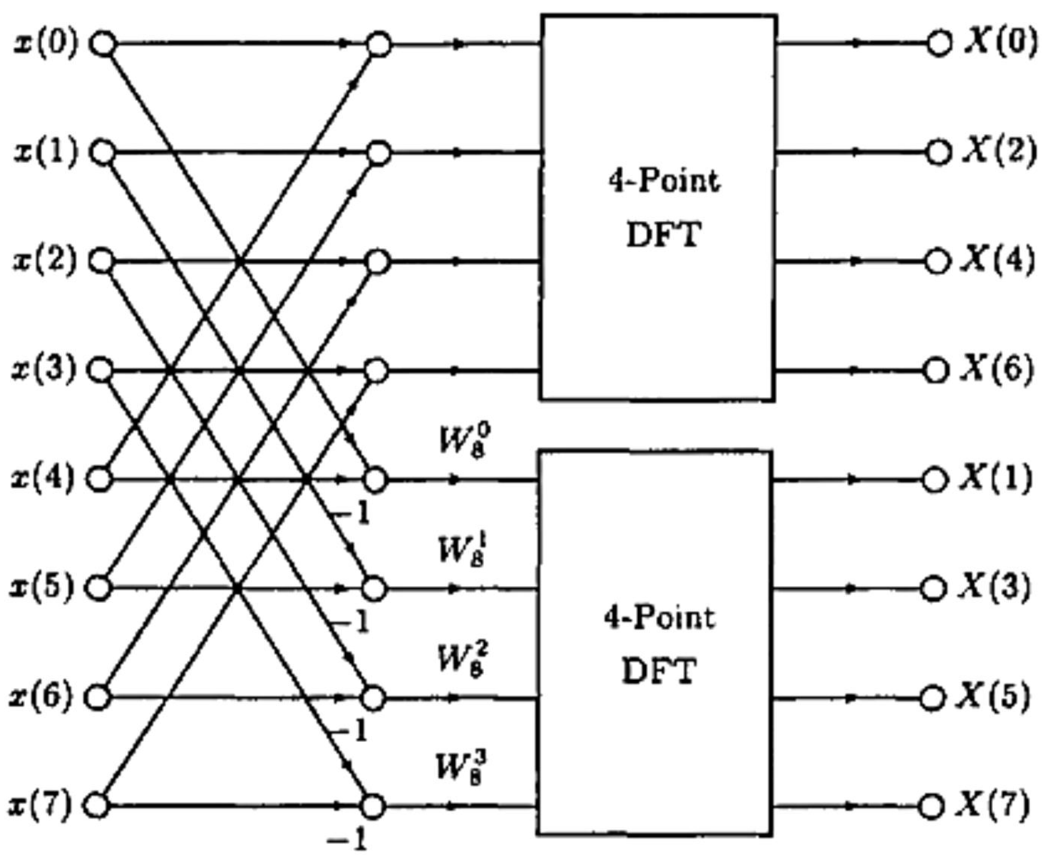


Fig.5 An 8-point DIF-FFT

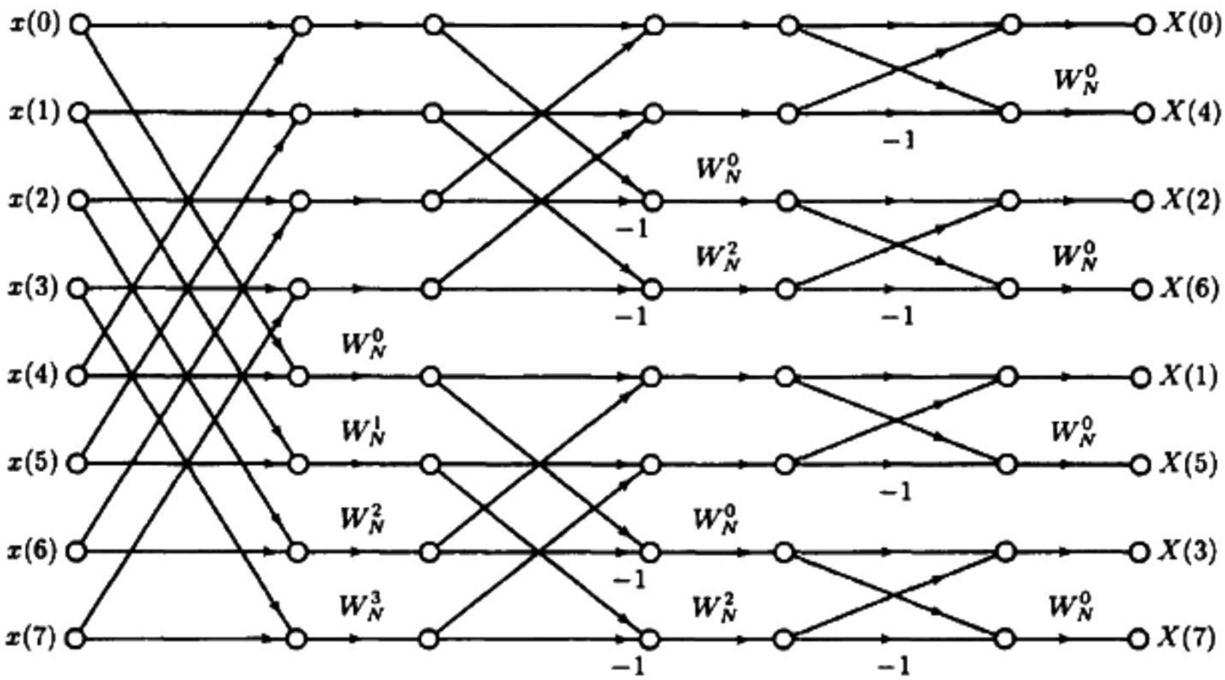


Fig.6 An Eight-point radix-2 DIF FFT

Example: Compute the 8-point DFT of the sequence x(n)=[0 1 2 3] using 8-point radix-2 DIF-FFT algorithm.

Sol :

$$x(n) = (0,1,2,3,0,0,0,0)$$

$$X(k) = \sum_{n=0}^{7} x(n) W_8^{kn}$$

$$X(0) = x(0)W_8^0 + x(1)W_8^0 + x(2)W_8^0 + x(3)W_8^0 = 6$$

$$X(1) = x(0)W_8^0 + x(1)W_8^1 + x(2)W_8^2 + x(3)W_8^3$$
$$= -1.414 - j4.828$$

$$X(2) = x(0)W_8^0 + x(1)W_8^2 + x(2)W_8^4 + x(3)W_8^6$$
$$= -2 + j2$$

$$X(3) = x(0)W_8^0 + x(1)W_8^3 + x(2)W_8^6 + x(3)W_8^9$$
$$= 1.414 - j0.828$$

⋮ ⋮ ⋮ ⋮ ⋮ ⋮

$$X(7) = x(0)W_8^0 + x(1)W_8^7 + x(2)W_8^{14} + x(3)W_8^{21} =$$
$$-1.414 + j4.828$$