جامعة المستقبل
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Discrete Mathematics

Lecture 1
The Foundations: Logic and Proofs

By

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The Foundations: Logic and Proofs

The rules of logic specify the meaning of mathematical statements. For instance, these rules help us understand and reason with statements such as “There exists an integer that is not the sum of two squares” and “For every positive integern, the sum of the positive integers not exceeding n is n(n+1)/2.” Logic is the basis of all mathematical reasoning, and of all automated reasoning. It has practical applications to the design of computing machines, to the specification of systems, to artificial intelligence, to computer programming, to programming languages, and to other areas of computer science, as well as to many other fields of study.

To understand mathematics, we must understand what makes up a correct mathematical argument, that is, a proof. Once we prove a mathematical statement is true, we call it a theorem. A collection of theorems on a topic organize what we know about this topic. To learn a mathematical topic, a person needs to actively construct mathematical arguments on this topic, and not just read exposition. Moreover, knowing the proof of a theorem often makes it possible to modify the result to fit new situations.

Propositional Logic

A Proposition Is a declarative sentence (that is, a sentence that declares a fact) that is either true or false, but not both.

For example, all the following declarative sentences are propositions.

1. Washington, D.C., is the capital of the United States of America.
2. Toronto is the capital of Canada.
3. 1+1=2.
4. 2+2=3.

Propositions 1 and 3 are true, whereas 2 and 4 are false. Some sentences that are not propositions such as:

1. What time is it?

2. Read this carefully.

3.x+1=2.

4.x+y=z.

Sentences 1 and 2 are not propositions because they are not declarative sentences. Sentences 3 and 4 are not propositions because they are neither true nor false. Note that each of sentences 3 and 4 can be turned into a proposition if we assign values to the variables. We use letters to denote propositional variables (or statement variables), that is, variables that represent propositions, just as letters are used to denote numerical variables. The conventional letters used for propositional variables are ***p, q, r, s, …***. The truth value of a proposition is true, denoted by **T**, if it is a true proposition, and the truth value of a proposition is false, denoted by **F**, if it is a false proposition.

Defination 1: Negation

Let p be a proposition. The *negation of* p, denoted by $\neg p$ (also denoted by $\overline{p}$), is the statement

"It is not the case that p."

The proposition $\neg p$ is read "not p." The truth value of the negation of p, $\neg p$, is the opposite of the truth value of p.

Example 1:

Find the negation of the proposition "Michael's PC runs Linux" and express this in simple English.

Solution:

The negation is "It is not the case that Michael's PC runs Linux."

This negation can be more simply expressed as "Michael's PC does not run Linux."

Example 2:

Find the negation of the proposition "Vandana's smartphone has at least 32GB of memory" and express this in simple English.

Solution:

The negation is

"It is not the case that Vandana's smartphone has at least 32GB of memory."

This negation can also be expressed as

"Vandana's smartphone does not have at least 32GB of memory"

or even more simply as

"Vandana's smartphone has less than 32GB of memory."

Table 1 displays the truth table for the negation of a proposition p. This table has a row for each of the two possible truth values of a proposition p. Each row shows the truth value of $\neg p$ corresponding to the truth value of p for this row.

TABLE 1 The Truth Table for the Negation of a Proposition.

p	$\neg p$
T	F
F	T

Defination 2: Conjunction

Let p and q be propositions. The *conjunction* of p and q, denoted by $p \land q$, is the proposition "p and q." The conjunction $p \land q$ is true when both p and q are true and is false otherwise.

Table 2 displays the truth table of $p \land q$. This table has a row for each of the four possible combinations of truth values of p and q. The four rows correspond to the pairs of truth values TT, TF, FT, and FF, where the first truth value in the pair is the truth value of p and the second truth value is the truth value of q.

TABLE 2 **The Truth Table for the Conjunction of Two Propositions.**		
p	q	$p \wedge q$
T	T	T
T	F	F
F	T	F
F	F	F

❖ Note that in logic the word "but" sometimes is used instead of "and" in a conjunction. For example, the statement "The sun is shining, but it is raining" is another way of saying "The sun is shining and it is raining."

Example 3:

Find the conjunction of the propositions p and q where p is the proposition "Ali's PC has more than 16 GB free hard disk space" and q is the proposition "The processor in Ali's PC runs faster than 1 GHz."

Solution:

The conjunction of these propositions, $p \wedge q$, is the proposition "Ali's PC has more than 16 GB free hard disk space, and the processor in Ali's PC runs faster than 1 GHz."

This conjunction can be expressed more simply as "Ali's PC has more than 16 GB free hard disk space, and its processor runs faster than 1 GHz."

❖ For this conjunction to be true, both conditions given must be true. It is false, when one or both of these conditions are false.

Defination 3: Disjunction

Let p and q be propositions. The *disjunction* of p and q, denoted by $p \vee q$, is the proposition "p or q." The disjunction $p \vee q$ is false when both p and q are false and is true otherwise.

Table 3 displays the truth table for $p \vee q$.

TABLE 3 **The Truth Table for the Disjunction of Two Propositions.**

p	q	$p \vee q$
T	T	T
T	F	T
F	T	T
F	F	F

The use of the connective or in a disjunction corresponds to one of the two ways the word or is used in English, namely, as an inclusive or. A disjunction is true when at least one of the two propositions is true.

Example 4:

What is the disjunction of the propositions $\boldsymbol{p}$ and $\boldsymbol{q}$ where $\boldsymbol{p}$ and $\boldsymbol{q}$ are the same propositions as in Example 3?

Solution:

The disjunction of $\boldsymbol{p}$ and $\boldsymbol{q}$, $\boldsymbol{p \vee q}$, is the proposition

“Rebecca’s PC has at least 16 GB free hard disk space, or the processor in Rebecca’s PC runs faster than 1 GHz.”

This proposition is true when Rebecca’s PC has at least 16 GB free hard disk space, when the PC’s processor runs faster than 1 GHz, and when both conditions are true. It is false when both of these conditions are false, that is, when Rebecca’s PC has less than 16 GB free hard disk space and the processor in her PC runs at 1 GHz or slower.

Defination 4: Exclusive Or

Let p and q be propositions. The *exclusive or* of p and q, denoted by $p \oplus q$, is the proposition that is true when exactly one of p and q is true and is false otherwise.

The truth table for the exclusive or of two propositions is displayed in Table 4

TABLE 4 **The Truth Table for the Exclusive Or of Two Propositions.**

p	q	$p \oplus q$
T	T	F
T	F	T
F	T	T
F	F	F

Defination 5: Conditional Statement

Let p and q be propositions. The *conditional statement* $p \to q$ is the proposition "if p, then q." The conditional statement $p \to q$ is false when p is true and q is false, and true otherwise. In the conditional statement $p \to q$, p is called the *hypothesis* (or *antecedent* or *premise*) and q is called the *conclusion* (or *consequence*).

The statement $\boldsymbol{p \to q}$ is called a conditional statement because $\boldsymbol{p \to q}$ asserts that $\boldsymbol{q}$ is true on the condition that $\boldsymbol{p}$ holds. A conditional statement is also called an **implication**. The truth table for the conditional statement $\boldsymbol{p \to q}$ is shown in Table 5. Note that the statement $\boldsymbol{p \to q}$ is true when both $\boldsymbol{p}$ and $\boldsymbol{q}$ are true and when $\boldsymbol{p}$ is false (no matter what truth value $\boldsymbol{q}$ has). Because conditional statements play such an essential role in mathematical reasoning, a variety of terminology is used to express $\boldsymbol{p \to q}$. You will encounter most if not all of the following ways to express this conditional statement:

"if p, then q"
"if p, q"
"p is sufficient for q"
"q if p"
"q when p"
"a necessary condition for p is q"
"q unless $\neg p$"
"p implies q"
"p only if q"
"a sufficient condition for q is p"
"q whenever p"
"q is necessary for p"
"q follows from p"

TABLE 5 The Truth Table for the Conditional Statement $p \to q$.

p	q	$p \to q$
T	T	T
T	F	F
F	T	T
F	F	T

A useful way to understand the truth value of a conditional statement is to think of an obligation or a contract. For example, the pledge many politicians make when running for office is "If I am elected, then I will lower taxes." If the politician is elected, voters would expect this politician to lower taxes. Furthermore, if the politician is not elected, then voters will not have any expectation that this person will lower taxes, although the person may have sufficient influence to cause those in power to lower taxes. It is only when the politician is elected but does not lower taxes that voters can say that the politician has broken the campaign pledge. This last scenario corresponds to the case when p is true but q is false in $p \to q$.

Example 5:

Let p be the statement "Maria learns discrete mathematics" and q the statement "Maria will find a good job." Express the statement $p \to q$ as a statement in English.

Solution:

From the definition of conditional statements, we see that when p is the statement "Maria learns discrete mathematics" and q is the statement "Maria will find a good job," $p \to q$ represents the statement

"If Maria learns discrete mathematics, then she will find a good job."

There are many other ways to express this conditional statement in English. Among the most natural of these are:

"Maria will find a good job when she learns discrete mathematics."

"For Maria to get a good job, it is sufficient for her to learn discrete mathematics."

and

"Maria will find a good job unless she does not learn discrete mathematics."

Defination 6: Biconditional Statement

Let p and q be propositions. The *biconditional statement* $p \leftrightarrow q$ is the proposition "p if and only if q." The biconditional statement $p \leftrightarrow q$ is true when p and q have the same truth values, and is false otherwise. Biconditional statements are also called *bi-implications*.

The truth table for $\boldsymbol{p} \leftrightarrow \boldsymbol{q}$ is shown in Table 6. Note that the statement $\boldsymbol{p} \leftrightarrow \boldsymbol{q}$ is true when both the conditional statements $\boldsymbol{p} \rightarrow \boldsymbol{q}$ and $\boldsymbol{q} \rightarrow \boldsymbol{p}$ are true and is false otherwise. That is why we use the words "if and only if" to express this logical connective and why it is symbolically written by combining the symbols $\rightarrow$ and $\leftarrow$.

TABLE 6 The Truth Table for the Biconditional $p \leftrightarrow q$.

p	q	$p \leftrightarrow q$
T	T	T
T	F	F
F	T	F
F	F	T

There are some other common ways to express $\boldsymbol{p} \leftrightarrow \boldsymbol{q}$:

"$\boldsymbol{p}$ is necessary and sufficient for $\boldsymbol{q}$"

"if $\boldsymbol{p}$ then $\boldsymbol{q}$, and conversely"

"$\boldsymbol{p}$ iff $\boldsymbol{q}$."

The last way of expressing the biconditional statement $\boldsymbol{p} \leftrightarrow \boldsymbol{q}$ uses the abbreviation "iff" for "if and only if." Note that $\boldsymbol{p} \leftrightarrow \boldsymbol{q}$ has exactly the same truth value as $(p \rightarrow q) \wedge (q \rightarrow p)$.

Example 6:

Let $\boldsymbol{p}$ be the statement "You can take the flight," and let $\boldsymbol{q}$ be the statement "You buy a ticket." Then $\boldsymbol{p} \leftrightarrow \boldsymbol{q}$ is the statement

"You can take the flight if and only if you buy a ticket."

This statement is true if p and q are either both true or both false, that is, if you buy a ticket and can take the flight or if you do not buy a ticket and you cannot take the flight. It is false when p and q have opposite truth values, that is, when you do not buy a ticket, but you can take the flight (such as when you get a free trip) and when you buy a ticket but you cannot take the flight (such as when the airline bumps you).

Truth Tables of Compound Propositions

We have now introduced four important logical connectives—conjunctions, disjunctions, conditional statements, and biconditional statements—as well as negations. We can use these connectives to build up complicated compound propositions involving any number of propositional variables. We can use truth tables to determine the truth values of these compound propositions, as Example 7 illustrates. We use a separate column to find the truth value of each compound expression that occurs in the compound proposition as it is built up. The truth values of the compound proposition for each combination of truth values of the propositional variables in it is found in the final column of the table.

Example 7:

Construct the truth table of the compound proposition

$$(p \lor \neg q) \rightarrow (p \land q)$$

Solution:

Because this truth table involves two propositional variables p and q, there are four rows in this truth table, one for each of the pairs of truth values TT, TF, FT, and FF. The first two columns are used for the truth values of p and q, respectively. In the third column we find the truth value of $\neg q$, needed to find the truth value of $p \lor \neg q$, found in the fourth column. The fifth column gives the truth value of $p \land q$. Finally, the truth value of $(p \lor \neg q) \rightarrow (p \land q)$ is found in the last column. The resulting truth table is shown in Table 7.

TABLE 7 **The Truth Table of $(p \vee \neg q) \rightarrow (p \wedge q)$.**

p	q	$\neg q$	$p \vee \neg q$	$p \wedge q$	$(p \vee \neg q) \rightarrow (p \wedge q)$
T	T	F	T	T	T
T	F	T	T	F	F
F	T	F	F	F	T
F	F	T	T	F	F

Precedence of Logical Operators

We can construct compound propositions using the negation operator and the logical operators defined so far. We will generally use parentheses to specify the order in which logical operators in a compound proposition are to be applied. For instance, $(p \vee q) \wedge (\neg r)$ is the conjunction of $p \vee q$ and $\neg r$. However, to reduce the number of parentheses, we specify that the negation operator is applied before all other logical operators. This means that $\neg p \wedge q$ is the conjunction of $\neg p$ and q, namely, $(\neg p) \wedge q$, not the negation of the conjunction of p and q, namely $\neg(p \wedge q)$.

Another general rule of precedence is that the conjunction operator takes precedence over the disjunction operator, so that $p \wedge q \vee r$ means $(p \wedge q) \vee r$ rather than $p \wedge (q \vee r)$. Because this rule may be difficult to remember, we will continue to use parentheses so that the order of the disjunction and conjunction operators is clear.

Finally, it is an accepted rule that the conditional and biconditional operators $\rightarrow$ and $\leftrightarrow$ have lower precedence than the conjunction and disjunction operators, $\wedge$ and $\vee$. Consequently, $p \vee q \rightarrow r$ is the same as $(p \vee q) \rightarrow r$. We will use parentheses when the order of the conditional operator and biconditional operator is at issue, although the conditional operator has precedence over the biconditional operator. Table 8 displays the precedence levels of the logical operators, $\neg$, $\wedge$, $\vee$, $\rightarrow$, and $\leftrightarrow$.

TABLE 8 Precedence of Logical Operators.	
Operator	*Precedence*
$\neg$	1
$\wedge$ $\vee$	2 3
$\rightarrow$ $\leftrightarrow$	4 5

Logic and Bit Operations

Computers represent information using bits. A bit is a symbol with two possible values, namely, 0 (zero) and 1 (one). This meaning of the word bit comes from binary digit, because zeros and ones are the digits used in binary representations of numbers. The well-known statistician John Tukey introduced this terminology in 1946. A bit can be used to represent a truth value, because there are two truth values, namely, true and false. As is customarily done, we will use a 1 bit to represent true and a 0 bit to represent false. That is, 1 represents T (true), 0 represents F (false). A variable is called a Boolean variable if its value is either true or false.

Truth Value	*Bit*
T	1
F	0

Consequently, a Boolean variable can be represented using a bit. Computer bit operations correspond to the logical connectives. By replacing true by a one and false by a zero in the truth tables for the operators$\wedge$, $\vee$, and$\oplus$, the tables shown in Table 9 for the corresponding bit operations are obtained. We will also use the notation OR, AND, and XOR for the operators$\vee$, $\wedge$, and $\oplus$, as is done in various programming languages.

TABLE 9 Table for the Bit Operators *OR, AND*, and *XOR*.

x	y	$x \vee y$	$x \wedge y$	$x \oplus y$
0	0	0	0	0
0	1	1	0	1
1	0	1	0	1
1	1	1	1	0

Defination 7: Bit String

A *bit string* is a sequence of zero or more bits. The *length* of this string is the number of bits in the string.

Information is often represented using bit strings, which are lists of zeros and ones. When this is done, operations on the bit strings can be used to manipulate this information.

Example 8:

101010011 is a bit string of length nine.

We can extend bit operations to bit strings. We define the bitwise OR, bitwise AND, and bitwise XOR of two strings of the same length to be the strings that have as their bits the OR, AND, and XOR of the corresponding bits in the two strings, respectively. We use the symbols $\vee$, $\wedge$, and $\oplus$to represent the bitwise OR, bitwise AND, and bitwise XOR operations, respectively. We illustrate bitwise operations on bit strings with Example 9.

Example 9:

Find the bitwise OR, bitwise AND, and bitwise XOR of the bit strings 01 1011 0110 and11 0001 1101.

Solution:

The bitwise OR, bitwise AND, and bitwise XOR of these strings are obtained by taking the OR, AND, and XOR of the corresponding bits, respectively. This gives us

	01 1011 0110	
	11 0001 1101	
	11 1011 1111	bitwise *OR*
	01 0001 0100	bitwise *AND*
	10 1010 1011	bitwise *XOR*

Homework 1

Which of these sentences are propositions? What are the truth values of those that are propositions?

a) Boston is the capital of Massachusetts.

b) Miami is the capital of Florida.

c) 2+3=5.

d)5+7=10.

e) x+2=11.

Homework 2

Let ***p*** and ***q*** be the propositions "The election is decided" and "The votes have been counted," respectively. Express each of these compound propositions as an English sentence.

a) $\neg p$ **b)** $p \vee q$
c) $\neg p \wedge q$ **d)** $q \rightarrow p$
e) $\neg q \rightarrow \neg p$ **f)** $\neg p \rightarrow \neg q$
g) $p \leftrightarrow q$ **h)** $\neg q \vee (\neg p \wedge q)$