CH-9-

Theory of vibrations : نظرية الاهتزازات

1. Longitudinal and Transverse Vibrations الاهتزازات الطولية والعرضية

i. Introduction : المقدمة

When elastic bodies such as a spring, a beam and a shaft are displaced from the equilibrium position by the application of external forces, and then released, they execute .

ii. vibratory motion: الحركة الذبذبية

This is due to the reason that, when a body is displaced, the internal forces in the form of elastic or strain energy are present in the body. At release, these forces bring the body to its original position. When the body reaches the equilibrium position, the whole of the elastic or strain energy is converted into kinetic energy due to which the body continues to move in the opposite direction. The whole of the kinetic energy is again converted into strain energy due to which the body again returns to the equilibrium position. In this way, the vibratory motion is repeated indefinitely.

iii. Terms Used in Vibratory Motion:المصطلحات الفنية للحركة الاهتزازية

The following terms are commonly used in connection with the vibratory motions :

1. **Period of vibration or time period.** زمن الذبذبة

It is the time interval after which the motion is repeated itself. The period of vibration is usually expressed in seconds.

2. **Cycle.** **الحركة الكاملة خلال ذبذبة**

It is the motion completed during one time period.

3. **Frequency.** **التردد**

It is the number of cycles described in one second. In S.I. units, the frequency is expressed in hertz (briefly written as Hz) which is equal to one cycle per second.

iv. Types of Vibratory Motion انواع الحركات الاهتزازية

The following types of vibratory motion are important from the subject point of view :

1. **Free or natural vibrations.** **الاهتزازات الطبيعية**

When no external force acts on the body, after giving it an initial displacement, then the body is said to be under free or natural vibrations. The frequency of the free vibrations is called free or natural frequency.

2. **Forced vibrations.** **اهتزازات القصرية (بقوه)**

When the body vibrates under the influence of external force, then the body is said to be under forced vibrations. The external force applied to the body is a periodic disturbing force created by unbalance. The vibrations have the same frequency as the applied force.

Note : **When the frequency of the external force is same as that of the natural vibrations, resonance takes place.**

3. Damped vibrations. **الاهتزاز المخمد**

When there is a reduction in amplitude over every cycle of vibration, the motion is said to be damped vibration. This is due to the fact that a certain amount of energy possessed by the vibrating system is always dissipated in overcoming frictional resistances to the motion.

4. resonace : **ظاهرة الرنين**

the resonace appear or take place , when the frequaency of external force = natural frequency of material .

v. Types of Free Vibrations انواع الاهتزازات الحره

The following three types of free vibrations are important from the subject point of view :

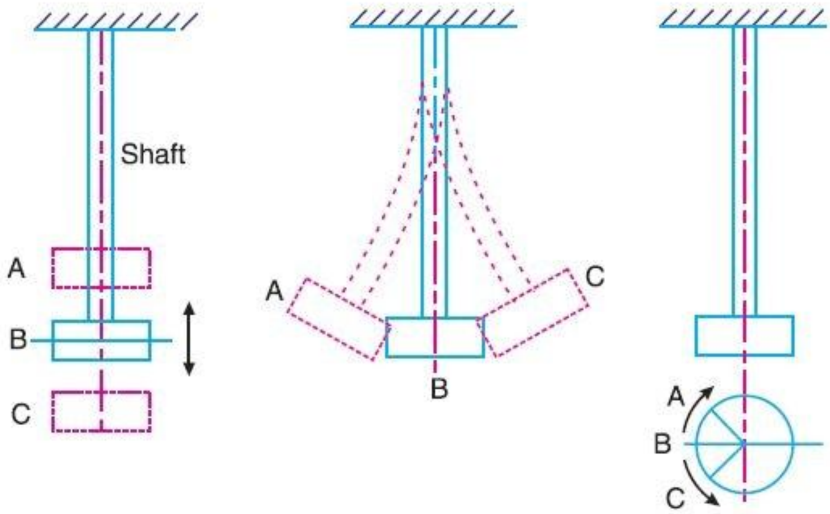
The following three types of free vibrations are important from the subject point of view :

1. Longitudinal vibrations,

2. Transverse vibrations, and

3. Torsional vibrations.

Consider a weightless constraint (spring or shaft) whose one end is fixed and the other end carrying a heavy disc, as shown in Fig..1. This system may execute one of the three above mentioned types of vibrations.



B = Mean position ; *A* and *C* = Extreme positions.
(*a*) Longitudinal vibrations. (*b*) Transverse vibrations. (*c*) Torsional vibrations.
Fig. 1. Types of free vibrations.

1. Longitudinal vibrations. الاهتزازات الحرة الطولية

When the particles of the shaft or disc moves parallel to the axis of the shaft, as shown in Fig. 1 (a), then the vibrations are known as longitudinal vibrations. In this case, the shaft is elongated and shortened alternately and thus the tensile and compressive stresses are induced alternately in the shaft.

2. Transverse vibrations. الاهتزازات الانتقالية

When the particles of the shaft or disc move approximately perpendicular to the axis of the shaft, as shown in Fig. 1 (b), then the vibrations are known as transverse vibrations. In this case, the shaft is straight and bent alternately and bending stresses are induced in the shaft.

3. Torsional vibrations*. الاهتزازات الالتوائية

When the particles of the shaft or disc move in a circle about the axis of the shaft, as shown in Fig.1 (c), then the vibrations are known as torsional vibrations. In this case, the shaft is twisted and untwisted alternately and the torsional shear stresses are induced in the shaft.

4. **Torsional vibrations*.** الاهتزازات الالتوائية

When the particles of the shaft or disc move in a circle about the axis of the shaft, as shown in Fig. .1 (c), then the vibrations are known as torsional vibrations. In this case, the shaft is twisted and untwisted alternately and the torsional shear stresses are induced in the shaft.

Note :

If the limit of proportionality (i.e. stress proportional to strain) is not exceeded in the three types of vibrations, then the restoring force in longitudinal and transverse vibrations or the restoring couple in torsional vibrations which is exerted on the disc by the shaft (due to the stiffness of the shaft) is directly proportional to the displacement of the disc from its equilibrium or mean position. Hence it follows that the acceleration towards the equilibrium position is directly proportional to the displacement from that position and the vibration is, therefore, simple harmonic.

vi. Natural Frequency of Free Longitudinal Vibrations

طرق حساب التردد الطبيعي للاهتزازات الحرة الطولية

The natural frequency of the free longitudinal vibrations may be determined by the following three methods :

1. Equilibrium Method طريقة التوازن

Consider a constraint (i.e. spring) of negligible mass in an unstrained position, as shown in Fig. 2 (a).

Let s = Stiffness of the constraint. It is the force required to produce unit displacement in the direction of vibration. It is usually expressed in N/m.

m = Mass of the body suspended from the constraint in kg,

W = Weight of the body in newtons = $m.g$,

δ = Static deflection of the spring in metres due to weight W newtons, and

x = Displacement given to the body by the external force, in metres.

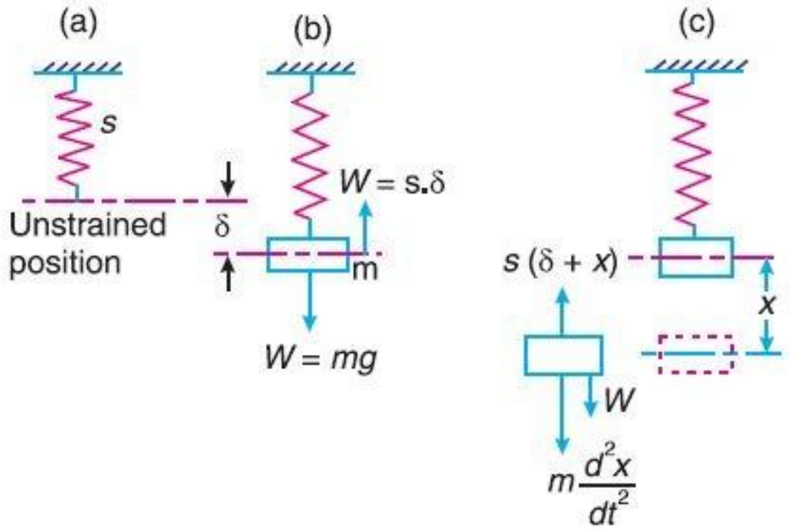


Fig. 2. Natural frequency of free longitudinal vibrations.

In the equilibrium position, as shown in Fig. 2 (*b*), the gravitational pull $W = m.g$, is balanced by a force of spring, such that $W = s.\delta$.

Since the mass is now displaced from its equilibrium position by a distance x, as shown in Fig. 23.2 (*c*), and is then released, therefore after time t,

Restoring force $= W - s(\delta + x) = W - s.\delta - s.x$

$= s.\delta - s.\delta - s.x = -s.x$ . . . ($\because W = s.\delta$) . . . (1)

. . . (Taking upward force as negative)

and Accelerating force = Mass × Acceleration

$= m \times \frac{d^2x}{dt^2}$ . . . (Taking downward force as positive) . . . (2)

Equating equations (1) and (2), the equation of motion of the body of mass m after time t is

$$m \times \frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \ldots (3)$$

We know that the fundamental equation of simple harmonic motion is

$$\frac{d^2x}{dt^2} + \omega^2.x = 0 \qquad \ldots (4)$$

Comparing equations (3) and (4), we have

$$\omega = \sqrt{\frac{s}{m}}$$

$\therefore$ Time period, $t_p = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{m}{s}}$

and natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$ . . . ($\because\ m.g = s.\delta$)

Taking the value of g as 9.81 m/s^2 and δ in metres,

$$f_n = \frac{1}{2\pi}\sqrt{\frac{9.81}{\delta}} = \frac{0.4985}{\sqrt{\delta}}\,\text{Hz}$$

Note : The value of static deflection δ may be found out from the given conditions of the problem. For longitudinal vibrations, it may be obtained by the relation,

$$\frac{\text{Stress}}{\text{Strain}} = E \quad \text{or} \quad \frac{W}{A} \times \frac{l}{\delta} = E \quad \text{or} \quad \delta = \frac{W.l}{E.A}$$

where δ = Static deflection *i.e.* extension or compression of the constraint,

W = Load attached to the free end of constraint,

l = Length of the constraint,

E = Young's modulus for the constraint, and

A = Cross-sectional area of the constraint.

2. Energy method طريقة الطاقة

We know that the kinetic energy is due to the motion of the body and the potential energy is with respect to a certain datum position which is equal to the amount of work required to move the body from the datum position. In the case of vibrations, the datum position is the mean or equilibrium position at which the potential energy of the body or the system is zero. In the free vibrations, no energy is transferred to the system or from the system. Therefore the summation of kinetic energy and potential energy must be a constant quantity which is same at all the times. In other words,

$$\therefore \quad \frac{d}{dt}(K.E. + P.E.) = 0$$

We know that kinetic energy,

$$K.E. = \frac{1}{2} \times m \left(\frac{dx}{dt} \right)^2$$

and potential energy,

$$P.E. = \left(\frac{0 + s.x}{2} \right) x = \frac{1}{2} \times s.x^2$$

. . . ($\because$ P.E. = Mean force × Displacement)

$$\therefore \quad \frac{d}{dt}\left[\frac{1}{2} \times m \left(\frac{dx}{dt} \right)^2 + \frac{1}{2} \times s.x^2 \right] = 0$$

$$\frac{1}{2} \times m \times 2 \times \frac{dx}{dt} \times \frac{d^2x}{dt^2} + \frac{1}{2} \times s \times 2x \times \frac{dx}{dt} = 0$$

or $\quad m \times \frac{d^2x}{dt^2} + s.x = 0 \quad$ or $\quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad$ . . . (Same as before)

The time period and the natural frequency may be obtained as discussed in the previous method.

3. Rayleigh's method طريقة رالي

In this method, the maximum kinetic energy at the mean position is equal to the maximum potential energy (or strain energy) at the extreme position. Assuming the motion executed by the vibration to be simple harmonic, then .

$$x = X \sin \omega.t \quad \ldots (1)$$

where x = Displacement of the body from the mean position after time t seconds, and

X = Maximum displacement from mean position to extreme position.

Now, differentiating equation (1), we have

$$\frac{dx}{dt} = \omega \times X \cos \omega.t$$

Since at the mean position, $t = 0$, therefore maximum velocity at the mean position,

$$v = \frac{dx}{dt} = \omega.X$$

∴ Maximum kinetic energy at mean position

$$= \frac{1}{2} \times m.v^2 = \frac{1}{2} \times m.\omega^2.X^2 \quad \ldots (2)$$

and maximum potential energy at the extreme position

$$= \left(\frac{0 + s.X}{2} \right) X = \frac{1}{2} \times s.X^2 \quad \ldots (3)$$

Equating equations (2) and (3),

$$\frac{1}{2} \times m.\omega^2.X^2 = \frac{1}{2} \times s.X^2 \quad \text{or} \quad \omega^2 = \frac{s}{m}, \text{ and } \omega = \sqrt{\frac{s}{m}}$$

∴ Time period,

$$t_p = \frac{2\pi}{\omega} = 2\pi \sqrt{\frac{s}{m}} \quad \ldots \text{(Same as before)}$$

and natural frequency,

$$f_n = \frac{1}{t_p} = \frac{\omega}{2\pi} = \frac{1}{2\pi} \sqrt{\frac{s}{m}} \quad \ldots \text{(Same as before)}$$

Note : In all the above expressions, ωis known as natural circular frequency and is generally denoted by ωn.

vii. Natural Frequency of Free Transverse Vibrations

التردد الطبيعي للاهتزاز العرضي

Consider a shaft of negligible mass, whose one end is fixed and the other end carries a body of weight W, as shown in Fig. 3.

Let s = Stiffness of shaft,

δ = Static deflection due to weight of the body,

x = Displacement of body from mean position after time t.

m = Mass of body = W/g

As discussed in the previous article,

Restoring force $= -s.x$ . . . (1)

and accelerating force $= m \times \frac{d^2x}{dt^2}$ . . . (2)

Equating equations (1) and (2), the equation of motion becomes

$$m \times \frac{d^2x}{dt^2} = -s.x \quad \text{or} \quad m \times \frac{d^2x}{dt^2} + s.x = 0$$

$$\therefore \quad \frac{d^2x}{dt^2} + \frac{s}{m} \times x = 0 \qquad \text{. . . (Same as before)}$$

Hence, the time period and the natural frequency of the transverse vibrations are same as that of longitudinal vibrations. Therefore

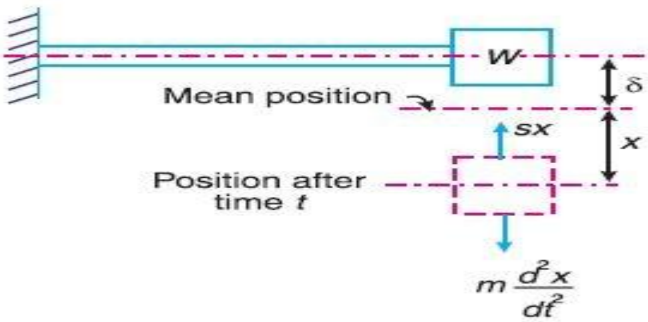


Fig. 3. Natural frequency of free transverse vibrations.

Hence, the time period and the natural frequency of the transverse vibrations are same as that of longitudinal vibrations. Therefore .

Time period, $t_p = 2\pi\sqrt{\frac{m}{s}}$

and natural frequency, $f_n = \frac{1}{t_p} = \frac{1}{2\pi}\sqrt{\frac{s}{m}} = \frac{1}{2\pi}\sqrt{\frac{g}{\delta}}$

Note : The shape of the curve, into which the vibrating shaft deflects, is identical with the static deflection curve of a cantilever beam loaded at the end. It has been proved in the text book on Strength of Materials, that the static deflection of a cantilever beam loaded at the free end is

$$\delta = \frac{Wl^3}{3EI} \text{ (in metres)}$$

where W = Load at the free end, in newtons,

l = Length of the shaft or beam in metres,

E = Young's modulus for the material of the shaft or beam in N/m^2, and

I = Moment of inertia of the shaft or beam in m^4.