Chapter four

Applications of derivatives

4-1- L'Hopital rule :

Suppose that $f(x_o) = g(x_o) = 0$ and that the functions f and g are both differentiable on an open interval (a, b) that contains the point x_o. Suppose also that $g'(x) \neq 0$ at every point in (a, b) except possibly x_o. Then :

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = \lim_{x\to x_0} \frac{f'(x)}{g'(x)} \quad \text{provided the limit exists .}$$

Differentiate f and g as long as you still get the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = x_o$. Stop differentiating as soon as you get something else . L'Hopital's rule does not apply when either the numerator or denominator has a finite non-zero limit .

EX-1 **– Evaluate the following limits :**

$$1)\ \lim_{x\to o} \frac{\sin x}{x} \qquad 2)\ \lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-4}$$

$$3)\ \lim_{x\to 0} \frac{x-\sin x}{x^3} \qquad 4)\ \lim_{x\to \frac{\pi}{2}} -(x-\frac{\pi}{2}).\tan x$$

Sol. **–**

$$1)\quad \lim_{x\to 0} \frac{\sin x}{x} \Rightarrow \frac{0}{0} \ \ using\ \ L'Hoptal's\ \ rule \Rightarrow$$

$$= \lim_{x\to 0} \frac{\cos x}{1} = \cos 0 = 1$$

$$2)\quad \lim_{x\to 2} \frac{\sqrt{x^2+5}-3}{x^2-4} \Rightarrow \frac{0}{0} \ \ using\ \ L'Hoptal's\ \ rule \Rightarrow$$

$$= \lim_{x\to 2} \frac{\frac{x}{\sqrt{x^2+5}}}{2x} = \lim_{x\to 2} \frac{1}{2\sqrt{x^2+5}} = \frac{1}{2\sqrt{4+5}} = \frac{1}{6}$$

$$3)\quad \lim_{x\to 0} \frac{x-\sin x}{x^3} \Rightarrow \frac{0}{0} \ \ using\ \ L'Hoptal's\ \ rule \Rightarrow$$

$$= \lim_{x\to 0} \frac{1-\cos x}{3x^2} \Rightarrow \frac{0}{0} \ \ using\ \ L'Hopital's\ \ rule \Rightarrow$$

$$= \frac{1}{6} \lim_{x\to 0} \frac{\sin x}{x} = \frac{1}{6}$$

$$4)\quad \lim_{x\to\frac{\pi}{2}} -(x-\frac{\pi}{2})\tan x \Rightarrow 0.\infty \text{ we can't using L'Hoptal's rule} \Rightarrow$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{x-\frac{\pi}{2}}{\cos x}.\lim_{x\to\frac{\pi}{2}} \sin x \Rightarrow \frac{0}{0} \text{ using L'Hopital's rule} \Rightarrow$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{1}{-\sin x}.\lim_{x\to\frac{\pi}{2}} \sin x = \frac{1}{\sin\frac{\pi}{2}}.\sin\frac{\pi}{2} = 1$$

4-2- <u>The slope of the curve</u> :

Secant to the curve is a line through two points on a curve.

Slopes and tangent lines :

1. **we start with what we can calculate , namely the slope of secant through *P* and a point *Q* nearby on the curve .**
2. **we find the limiting value of the secant slope (if it exists) as *Q* approaches *p* along the curve .**
3. **we take this number to be the slope of the curve at *P* and define the tangent to the curve at *P* to be the line through *p* with this slope .**

The derivative of the function *f* is the slope of the curve :

$$\text{the slope} = m = f'(x) = \frac{dy}{dx}$$

***<u>EX-2</u>*- Write an equation for the tangent line at *x = 3* of the curve :**

$$f(x) = \frac{1}{\sqrt{2x+3}}$$

***<u>Sol.</u>*-**

$$m = f'(x) = -\frac{1}{\sqrt{(2x+3)^3}} \Rightarrow [m]_{x=3} = f'(3) = -\frac{1}{27}$$

$$f(3) = \frac{1}{\sqrt{2*3+3}} = \frac{1}{3}$$

The equation of the tangent line is :

$$y - \frac{1}{3} = -\frac{1}{27}(x-3) \Rightarrow 27y + x = 12$$

4-3- Velocity and acceleration and other rates of changes :

- The average velocity of a body moving along a line is :

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{displacement}{time\ travelled}$$

The instantaneous velocity of a body moving along a line is the derivative of its position $s = f(t)$ with respect to time t .

i.e. $$v = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

- The rate at which the particle's velocity increase is called its acceleration a . If a particle has an initial velocity v and a constant acceleration a, then its velocity after time t is $v + at$.

$$average\ acceleration = a_{av} = \frac{\Delta v}{\Delta t}$$

The acceleration at an instant is the limit of the average acceleration for an interval following that instant , as the interval tends to zero .

i.e. $$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

- The average rate of a change in a function $y = f(x)$ over the interval from x to $x + \Delta x$ is :

$$average\ rate\ of\ change = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The instantaneous rate of change of f at x is the derivative.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ provided the limit exists .

EX-3- The position s (in meters) of a moving body as a function of time t (in second) is : $s = 2t^2 + 5t - 3$; find :

a) The displacement and average velocity for the time interval from $t = 0$ to $t = 2$ seconds .

b) The body's velocity at $t = 2$ seconds .

Sol.-

$$a)\quad 1)\qquad \Delta s = s(t+\Delta t) - s(t) = 2(t+\Delta t)^2 + 5(t+\Delta t) - 3 - \left[2t^2 + 5t - 3\right]$$
$$= (4t+5)\Delta t + 2(\Delta t)^2$$
$$at\ t = 0\ and\ \Delta t = 2 \Rightarrow \Delta s = (4*0+5)*2 + 2*2^2 = 18$$
$$2)\quad v_{av} = \frac{\Delta s}{\Delta t} = \frac{(4t+5)\Delta t + 2(\Delta t)^2}{\Delta t} = 4t + 5 + 2.\Delta t$$
$$at\ t = 0\ \ and\ \Delta t = 2 \Rightarrow v_{av} = 4*0+5+2*2 = 9$$
$$b)\quad v(t) = \frac{d}{dt} f(t) = 4t+5$$
$$v(2) = 4*2+5 = 13$$

EX-4- A particle moves along a straight line so that after t (seconds) , its distance from O a fixed point on the line is s (meters) , where $s = t^3 - 3t^2 + 2t$:

i) when is the particle at O ?
ii) what is its velocity and acceleration at these times ?
iii) what is its average velocity during the first second ?
iv) what is its average acceleration between $t = 0$ and $t = 2$?

Sol. –

$$i)\quad at\ \ s = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-1)(t-2) = 0$$
$$either\ \ t = 0\ \ or\ \ t = 1\ \ or\ \ t = 2\ \ sec.$$
$$ii)\quad velocity = v(t) = 3t^2 - 6t + 2 \Rightarrow v(0) = 2m/s$$
$$\Rightarrow v(1) = -1m/s$$
$$\Rightarrow v(2) = 2m/s$$
$$acceleration = a(t) = 6t - 6 \Rightarrow a(0) = -6m/s^2$$
$$\Rightarrow a(1) = 0m/s^2$$
$$\Rightarrow a(2) = 6m/s^2$$
$$iii)\quad v_{av} = \frac{\Delta s}{\Delta t} = \frac{s(1)-s(0)}{1-0} = \frac{1-3+2-0}{1} = 0m/s$$
$$iv)\quad a_{av} = \frac{\Delta v}{\Delta t} = \frac{v(2)-v(0)}{2-0} = \frac{2-2}{2} = 0m/s^2$$

4-4- <u>Maxima and Minima</u> :

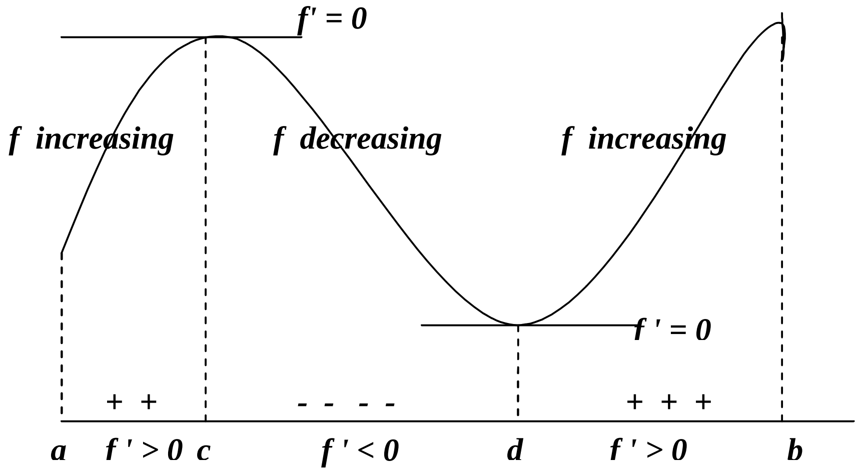
<u>Increasing and decreasing function</u> : **Let f be defined on an interval and x_1 , x_2 denoted a number on that interval :**

- **If $f(x_1) < f(x_2)$ when ever $x_1 < x_2$ then f is increasing on that interval .**
- **If $f(x_1) > f(x_2)$ when ever $x_1 < x_2$ then f is decreasing on that interval .**
- **If $f(x_1) = f(x_2)$ for all values of x_1 , x_2 then f is constant on that interval .**

<u>The first derivative test for rise and fall</u> : **Suppose that a function f has a derivative at every point x of an interval I. Then :**

- **f increases on I if $f'(x) > o, \quad \forall x \in I$**
- **f decreases on I if $f'(x) > o, \quad \forall x \in I$**

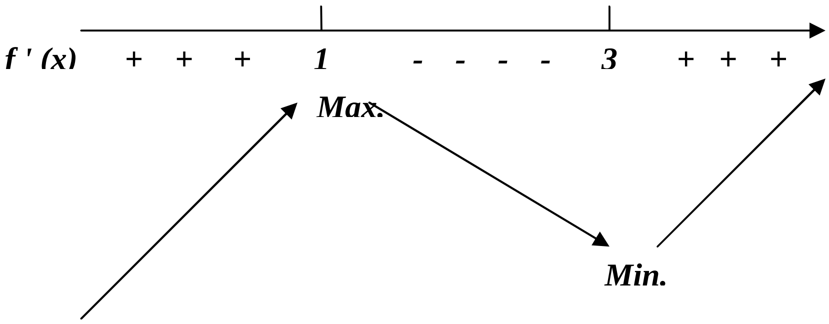
If f' changes from positive to negative values as x passes from left to right through a point c , then the value of f at c is a local maximum value of f , as shown in below figure . That is $f(c)$ is the largest value the function takes in the immediate neighborhood at $x = c$.



Similarly , if f' changes from negative to positive values as x passes left to right through a point d , then the value of f at d is a local minimum value of f . That is $f(d)$ is the smallest value of f takes in the immediate neighborhood of d .

<u>EX-5</u> **– Graph the function :** $y = f(x) = \frac{x^3}{3} - 2x^2 + 3x + 2$.

<u>Sol.</u>- $f'(x) = x^2 - 4x + 3 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,3$

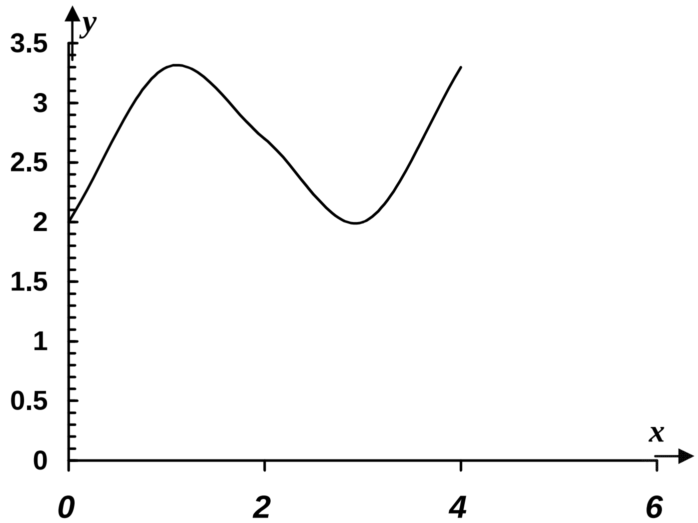


The function has a local maximum at $x = 1$ and a local minimum at $x = 3$.

To get a more accurate curve , we take :

x	*0*	*1*	*2*	*3*	*4*
f(x)	*2*	*3.3*	*2.7*	*2*	*3.3*

Then the graph of the function is :



<u>Concave down and concave up</u> : **The graph of a differentiable function $y = f(x)$ is concave down on an interval where f' decreases , and concave up on an interval where f' increases.**

<u>The second derivative test for concavity</u> : **The graph of $y = f(x)$ is concave down on any interval where $y'' < 0$, concave up on any interval where $y'' > 0$.**

<u>Point of inflection</u> : **A point on the curve where the concavity changes is called a point of inflection . Thus , a point of inflection on a twice – differentiable curve is a point where y'' is positive on one side and negative on other , i.e. $y'' = 0$.**

EX-6 **– Sketch the curve :** $y = \frac{1}{6}(x^3 - 6x^2 + 9x + 6)$ **.**

Sol. **-**

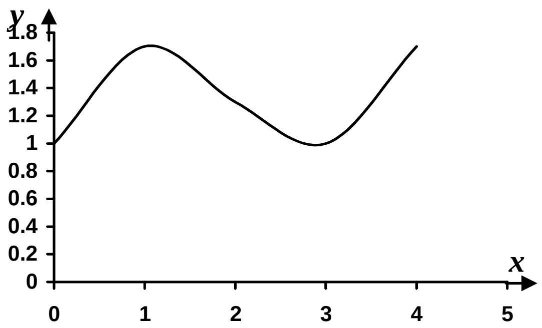
$$y' = \frac{1}{2}x^2 - 2x + \frac{3}{2} = 0 \Rightarrow x^2 - 4x + 3 = 0 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,3$$

$$y'' = x - 2 \Rightarrow at\ x = 1 \Rightarrow y'' = 1 - 2 = -1 < 0\ concave\ down\ .$$

$$\Rightarrow at\ x = 3 \Rightarrow y'' = 3 - 2 > 0 \qquad concave\ up\ .$$

$$\Rightarrow at\ y'' = 0 \Rightarrow x - 2 = 0 \Rightarrow x = 2\ \ point\ of\ inflection\ .$$

x	*0*	*1*	*2*	*3*	*4*
y	*1*	*1.7*	*1.3*	*1*	*1.7*



EX-7 **– What value of *a* makes the function :**

$f(x) = x^2 + \frac{a}{x}$ **, have :**

i) a local minimum at $x = 2$ **?**

ii) a local minimum at $x = -3$ **?**

iii) a point of inflection at $x = 1$ **?**

iv) show that the function can't have a local maximum for any value of a **.**

Sol. **–**

$$f(x) = x^2 + \frac{a}{x} \Rightarrow \frac{df}{dx} = 2x - \frac{a}{x^2} = 0 \Rightarrow a = 2x^3\ and\ \frac{d^2y}{dx^2} = 2 + \frac{2a}{x^3}$$

$$i)\quad at\ x = 2 \Rightarrow a = 2*8 = 16\ and\ \frac{d^2 f}{dx^2} = 2 + \frac{2*16}{2^3} = 6 > 0\ Mini.$$

$$ii)\quad at\ x = -3 \Rightarrow a = 2(-3)^3 = -54\ and\ \frac{d^2 f}{dx^2} = 2 + \frac{2(-54)}{(-3)^3} = 6 > 0\ Mini.$$

$$iii)\quad at\ x = 1 \Rightarrow \frac{d^2 f}{dx^2} = 2 + \frac{2a}{1} = 0 \Rightarrow a = -1$$

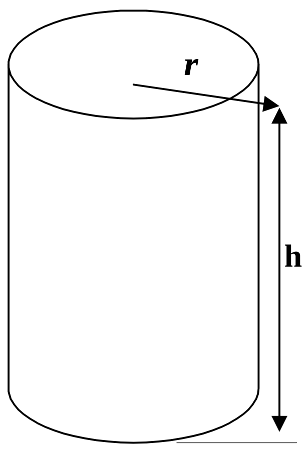
$$iv)\quad a = 2x^3 \Rightarrow \frac{d^2 f}{dx^2} = 2 + \frac{2(2x^3)}{x^3} = 6 > 0$$

Since $\frac{d^2 f}{dx^2} > 0$ *for all value of* x *in* $a = 2x^3$.

Hence the function don't have a local maximum .

EX-8 **– What are the best dimensions (use the least material) for a tin can which is to be in the form of a right circular cylinder and is to hold *1* gallon (*231* cubic inches) ?**

Sol. **– The volume of the can is :**



$$v = \pi r^2 h = 231 \Rightarrow h = \frac{231}{\pi r^2}$$

where *r* is radius , *h* is height .

The total area of the outer surface (top, bottom , and side) is :

$$A = 2\pi r^2 + 2\pi r h = 2\pi r^2 + 2\pi r \frac{231}{\pi r^2} \Rightarrow A = 2\pi r^2 + \frac{462}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{462}{r^2} = 0 \Rightarrow r = 3.3252\ inches$$

$$\frac{d^2 A}{dr^2} = 4\pi + \frac{924}{r^3} = 4\pi + \frac{924}{(3.3252)^3} = 37.714 > 0 \Rightarrow min.$$

$$h = \frac{231}{\pi r^2} = \frac{231}{\frac{22}{7}(3.3252)^2} = 6.6474\ inches$$

The dimensions of the can of volume *1* gallon have minimum surface area are :

r = 3.3252 **in. and *h = 6.6474*** **in.**

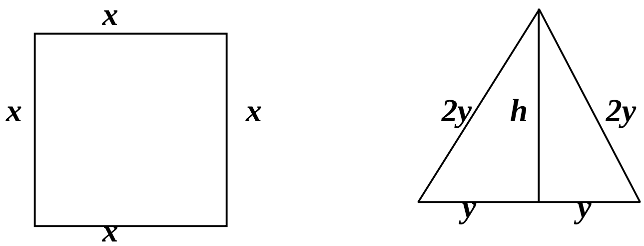
EX-9 **– A wire of length L is cut into two pieces , one being bent to form a square and the other to form an equilateral triangle . How should the wire be cut :**

a) if the sum of the two areas is minimum.

b) if the sum of the two areas is maximum.

Sol. : **Let x is a length of square.**

$2y$ is the edge of triangle .



The perimeter is $p = 4x + 6y = L \Rightarrow x = \frac{1}{4}(L - 6y)$.

$(2y)^2 = y^2 + h^2 \Rightarrow h = \sqrt{3}y$ *from triangle .*

The total area is $A = x^2 + yh = \frac{1}{16}(L - 6y)^2 + y\sqrt{3}y$

$$\Rightarrow A = \frac{1}{16}(L - 6y)^2 + \sqrt{3}y^2$$

$\frac{dA}{dy} = \frac{-3}{4}(L - 6y) + 2\sqrt{3}y = 0 \Rightarrow y = \frac{3L}{18 + 8\sqrt{3}}$

$\frac{d^2A}{dy^2} = \frac{9}{2} + 2\sqrt{3} > 0 \Rightarrow min.$

a) To minimized total areas cut for triangle $6y = \frac{9L}{9 + 4\sqrt{3}}$

And for square $L - \frac{9L}{9 + 4\sqrt{3}} = \frac{4\sqrt{3}L}{9 + 4\sqrt{3}}$.

b) To maximized the value of A on endpoints of the interval

$0 \le 4x \le L \Rightarrow 0 \le x \le \frac{L}{4}$

$at\ x = 0 \Rightarrow y = \frac{L}{6}\ and\ h = \frac{L}{2\sqrt{3}} \Rightarrow A_1 = \frac{L^2}{12\sqrt{3}}$

$at\ \ x = \frac{L}{4} \Rightarrow y = 0 \Rightarrow A_2 = \frac{L^2}{16}$

Since $A_2 = \frac{L^2}{16} > A_1 = \frac{L^2}{12\sqrt{3}}$

Hence the wire should not be cut at all but should be bent into a square .

Problems – 4

1. Find the velocity v if a particle's position at time t is $s = 180t – 16t^2$
When does the velocity vanish ? *(ans.: 5.625)*

2. If a ball is thrown straight up with a velocity of 32 ft./sec. , its high after t sec. is given by the equation $s = 32t - 16t^2$. At what instant will the ball be at its highest point ? and how high will it rise ?
(ans.: 1, 16)

3. A stone is thrown vertically upwards at 35 m./sec. . Its height is : $s = 35t – 4.9t^2$ in meter above the point of projection where t is time in second later :
a) What is the distance moved, and the average velocity during the 3^{rd} sec. (from $t = 2$ to $t = 3$) ?
b) Find the average velocity for the intervals $t = 2$ to $t = 2.5$, $t = 2$ to $t = 2.1$; $t = 2$ to $t = 2 + h$.
c) Deduce the actual velocity at the end of the 2^{nd} sec. .
(ans.: a) 10.5 , 10.5 ; b) 12.95, 14.91, 15.4-4.9h , c) 15.4)

4. A stone is thrown vertically upwards at 24.5 m./sec. from a point on the level with but just beyond a cliff ledge . Its height above the ledge t sec. later is $4.9t (5 – t)$ m. . If its velocity is v m./sec. , differentiate to find v in terms of t :
i) when is the stone at the ledge level ?
ii) find its height and velocity after 1 , 2 , 3 , and 6 sec. .
iii) what meaning is attached to negative value of s ? a negative value of v ?
iv) when is the stone momentarily at rest ? what is the greatest height reached ?
v) find the total distance moved during the 3^{rd} sec. .
(ans.:v=24.5-9.8t; i)0,5; ii)19.6,29.4,29.4,-29.4;14.7,4.9, -4.9,-34.3; iv)2.5;30.625; v)2.45)

5. A stone is thrown vertically downwards with a velocity of 10 m./sec. , and gravity produces on it an acceleration of 9.8 m./sec.^{2} :
a) what is the velocity after 1 , 2 , 3 , t sec. ?
b) sketch the velocity –time graph . *(ans.: 19.8, 29.6, 39.4,10+9.8t)*

6. A car accelerates from 5 km./h. to 41 km./h. in 10 sec. . Express this acceleration in : i)km./h. per sec. ii) m./sec.^{2}, iii) km./h.^{2} .
(ans.: i)3.6; ii)1; iii) 12960)

7. A car can accelerate at *4* m./sec.^{2} . How long will it take to reach *90* km./h. from rest ? ***(ans.: 6.25)***

8. An express train reducing its velocity to *40* km./h. , has to apply the brakes for *50* sec. . If the retardation produced is *0.5* m./sec.^{2} , find its initial velocity in km./h. . ***(ans.: 130)***

9. At the instant from which time is measured a particle is passing through *O* and traveling towards *A* , along the straight line *OA*. It is s m. from *O* after *t* sec. where $s = t(t-2)^2$:

i) when is it again at *O* ?

ii) when and where is it momentarily at rest ?

iii) what is the particle's greatest displacement from *O* , and how far does it moves , during the first *2* sec. ?

iv) what is the average velocity during the 3^{rd} sec. ?

v) at the end of the 1^{st} sec. where is the particle, which way is it going , and is its speed increasing or decreasing ?

vi) repeat (v) for the instant when $t = -1$.

(ans.:i)2;ii)0,32/27;iii)64/27;iv)3;v)OA;inceasing; vi)AO;decreasing)

10. A particle moves in a straight line so that after t sec. it is *s* m. , from a fixed point *O* on the line , where $s = t^4 + 3t^2$. Find :

i) The acceleration when $t = 1$, $t = 2$, and $t = 3$.

ii) The average acceleration between $t = 1$ and $t = 3$.

(ans.: i)18, 54,114; ii)58)

11. A particle moves along the x-axis in such away that its distance *x* cm. from the origin after *t* sec. is given by the formula $x = 27t - 2t^2$ what are its velocity and acceleration after *6.75* sec. ? How long does it take for the velocity to be reduced from *15* cm./sec. to *9* cm./sec., and how far does the particle travel mean while ? ***(ans.: 0,-4,1.5 ;18)***

12. A point moves along a straight line *OX* so that its distance *x* cm. from the point *O* at time *t* sec. is given by the formula

$x = t^3 - 6t^2 + 9t$. Find :

i) at what times and in what positions the point will have zero velocity .

ii) its acceleration at these instants .

iii) its velocity when its acceleration is zero .

(ans.: i)1,3;4,0; ii)-6,6; iii)-3)

13. A particle moves in a straight line so that its distance x cm. from a fixed point O on the line is given by $x = 9t^2 - 2t^3$ where t is the time in seconds measured from O . Find the speed of the particle when $t= 3$. Also find the distance from O of the particle when $t = 4$, and show that it is then moving towards O . *(ans.: 0, 16)*

14. Find the limits for the following functions by using L'Hopital's rule :

1) $\lim\limits_{x\to\infty}\dfrac{5x^2-3x}{7x^2+1}$ 2) $\lim\limits_{t\to 0}\dfrac{\sin t^2}{t}$

3) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{2x-\pi}{\cos x}$ 4) $\lim\limits_{t\to 0}\dfrac{\cos t-1}{t^2}$

5) $\lim\limits_{x\to\frac{\pi}{2}}\dfrac{1-\sin x}{1+\cos 2x}$ 6) $\lim\limits_{x\to\frac{\pi}{4}}\dfrac{\sin x-\cos x}{x-\frac{\pi}{4}}$

7) $\lim\limits_{x\to 1}\dfrac{2x^2-(3x+1)\sqrt{x}+2}{x-1}$ 8) $\lim\limits_{x\to 0}\dfrac{x(\cos x-1)}{\sin x-x}$

9) $\lim\limits_{x\to 0} x.\csc^2\sqrt{2x}$ 10) $\lim\limits_{x\to 0}\dfrac{\sin x^2}{x.\sin x}$

$$(ans.: 1)\frac{5}{7};2)0;3)-2;4)-\frac{1}{2};5)\frac{1}{4};6)\sqrt{2};7)-1;8)3;9)\frac{1}{2};10)1)$$

15. Find any local maximum and local minimum values , then sketch each curve by using first derivative :

1) $f(x)= x^3 - 4x^2 + 4x + 5$ $(ans.: max.(0.7,6.2);min.(2,5))$

2) $f(x)=\dfrac{x^2-1}{x^2+1}$ $(ans.: min.(0,-1))$

3) $f(x)= x^5-5x-6$ $(ans.: max.(-1,-2);min.(1,-10))$

4) $f(x)= x^{\frac{4}{3}}-x^{\frac{1}{3}}$ $(ans.: min.(0.25,-0.47))$

16. Find the interval of x-values on which the curve is concave up and concave down , then sketch the curve :

1) $f(x)=\dfrac{x^3}{3}+x^2-3x$ $(ans.: up(-1,\infty);down(-\infty,-1))$

2) $f(x)= x^2-5x+6$ $(ans.: up(-\infty,\infty))$

3) $f(x)= x^3 - 2x^2 + 1$ $(ans.: up(\frac{2}{3},\infty);down(-\infty,\frac{2}{3}))$

4) $f(x)= x^4-2x^2$ $(ans.: up(-\infty,-\frac{1}{\sqrt{3}}),(\frac{1}{\sqrt{3}},\infty);down(-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}))$

17. Sketch the following curve by using second derivative :

1) $y = \dfrac{x}{1+x^2}$ *(ans.: max.(1,0.5); min.(-1,-0.5))*

2) $y = -x(x-7)^2$ *(ans.: max.(7,0); min.(2.3,-50.8))*

3) $y = (x+2)^2(x-3)$ *(ans.: max.(-2,0); min.(1.3,-18.5))*

4) $y = x^2(5-x)$ *(ans.: max.(3.3,18.5); min.(0,0))*

18. What is the smallest perimeter possible for a rectangle of area *16* in.^{2} ? *(ans.: 16)*

19. Find the area of the largest rectangle with lower base on the x-axis and upper vertices on the parabola $y = 12 - x^2$. *(ans.:32)*

20) A rectangular plot is to be bounded on one side by a straight river and enclosed on the other three sides by a fence . With *800* m. of fence at your disposal . What is the largest area you can enclose ? *(ans.:80000)*

21) Show that the rectangle that has maximum area for a given perimeter is a square .

22) A wire of length *L* is available for making a circle and a square . How should the wire be divided between the two shapes to maximize the sum of the enclosed areas?
(ans.: all bent into a circle)

23) A closed container is made from a right circular cylinder of radius *r* and height *h* with a hemispherical dome on top . Find the relationship between *r* and *h* that maximizes the volume for a given surface area *s* . $\left(ans.: r = h = \sqrt{\dfrac{s}{5\pi}}\right)$

24) An open rectangular box is to be made from a piece of cardboard *8* in. wide and *15* in. long by cutting a square from each corner and bending up the sides Find the dimensions of the box of largest volume . *(ans.: height=5/3; width=14/3; length=35/3)*

Chapter five

Integration

5-1- <u>Indefinite integrals</u> :

The set of all anti derivatives of a function is called indefinite integral of the function.

Assume *u* and *v* denote differentiable functions of *x*, and *a*, *n*, and *c* are constants, then the integration formulas are:-

$$1)\ \int du = u(x) + c$$

$$2)\ \int a \cdot u(x)dx = a\int u(x)dx$$

$$3)\ \int \left(u(x) \mp v(x)\right)dx = \int u(x)\,dx \mp \int v(x)\,dx$$

$$4)\ \int u^n du = \frac{u^{n+1}}{n+1} + c \quad when \quad n \neq -1 \quad \& \quad \int u^{-1}du = \int \frac{1}{u}du = \ln u + c$$

$$5)\ \int a^u du = \frac{a^u}{\ln a} + c \quad \Rightarrow \quad \int e^u du = e^u + c$$

EX-1 **– Evaluate the following integrals:**

$$1)\ \int 3x^2\,dx \qquad 6)\ \int \frac{x+3}{\sqrt{x^2+6x}}\,dx$$

$$2)\ \int \left(\frac{1}{x^2} + x\right)dx \qquad 7)\ \int \frac{x+2}{x^2}\,dx$$

$$3)\ \int x\sqrt{x^2+1}\,dx \qquad 8)\ \int \frac{e^x}{1+3e^x}\,dx$$

$$4)\ \int \left(2t + t^{-1}\right)^2 dt \qquad 9)\ \int 3x^3 \cdot e^{-2x^4}\,dx$$

$$5)\ \int \sqrt{(z^2 - z^{-2})^2 + 4}\,dz \qquad 10)\ \int 2^{-4x}\,dx$$

Sol. **–**

$$1)\ \int 3x^2\,dx = 3\int x^2 dx = 3\frac{x^3}{3} + c = x^3 + c$$

$$2)\ \left(x^{-2}+x\right)dx=\int x^{-2}\,dx+\int x\,dx=\frac{x^{-1}}{-1}+\frac{x^2}{2}+c=-\frac{1}{x}+\frac{x^2}{2}+c$$

$$3)\ \int x\sqrt{x^2+1}\,dx=\frac{1}{2}\int 2x(x^2+1)^{1/2}dx=\frac{1}{2}\frac{(x^2+1)^{3/2}}{3/2}+c=\frac{1}{3}\sqrt{(x^2+1)3}+c$$

$$4)\ \int\left(2t+t^{-1}\right)^2 dt=\int\left(4t^2+4+t^{-2}\right)dt=4\frac{t^3}{3}+4t+\frac{t^{-1}}{-1}+c=\frac{4}{3}t^3+4t-\frac{1}{t}+c$$

$$5)\ \int\sqrt{(z^2-z^{-2})^2+4}\,dz=\int\sqrt{z^4-2+z^{-4}+4}\,dz=\int\sqrt{z^4+2+z^{-4}}\,dz$$

$$=\int\sqrt{(z^2+z^{-2})^2}\,dz=\int(z^2+z^{-2})\,dz=\frac{z^3}{3}+\frac{z^{-1}}{-1}+c=\frac{1}{3}z^3-\frac{1}{z}+c$$

$$6)\ \int\frac{x+3}{\sqrt{x^2+6x}}dx=\frac{1}{2}\int(2x+6)\cdot(x^2+6x)^{-1/2}dx$$

$$=\frac{1}{2}\cdot\frac{(x^2+6x)^{1/2}}{1/2}+c=\sqrt{x^2+6x}+c$$

$$7)\ \int\frac{x+2}{x^2}dx=\int\left(\frac{x}{x^2}+\frac{2}{x^2}\right)dx=\int\left(x^{-1}+2x^{-2}\right)dx=\ln x+\frac{2x^{-1}}{-1}+c=\ln x-\frac{2}{x}+c$$

$$8)\ \int\frac{e^x}{1+3e^x}dx=\frac{1}{3}\int 3e^x(1+3e^x)^{-1}dx=\frac{1}{3}\ln(1+3e^x)+c$$

$$9)\ \int 3x^3\cdot e^{-2x^4}dx=-\frac{3}{8}\int -8x^3\cdot e^{-2x^4}dx=-\frac{3}{8}\cdot e^{-2x^4}+c$$

$$10)\ \int 2^{-4x}dx=-\frac{1}{4}\int 2^{-4x}\cdot(-4dx)=-\frac{1}{4}\cdot 2^{-4x}\cdot\frac{1}{\ln 2}+c$$

5-2- Integrals of trigonometric functions :

The integration formulas for the trigonometric functions are:

$$6)\ \int\sin u\cdot du=-\cos u+c$$

$$7)\ \int\cos u\cdot du=\sin u+c$$

$$8)\ \int\tan u\cdot du=-\ln|\cos u|+c$$

$$9)\ \int\cot u\cdot du=\ln|\sin u|+c$$

$$10)\ \int\sec u\cdot du=\ln|\sec u+\tan u|+c$$

$$11)\ \int\csc u\cdot du=-\ln|\csc u+\cot u|+c$$

$$12)\ \int\sec^2 u\cdot du=\tan u+c$$

$$13)\ \int\csc^2 u\cdot du=-\cot u+c$$

$$14)\ \int\sec u\cdot\tan u\cdot du=\sec u+c$$

$$15)\ \int\csc u\cdot\cot u\cdot du=-\csc u+c$$

EX-2- **Evaluate the following integrals:**

1) $\int cos(3\theta - 1)d\theta$

2) $\int x \cdot sin(2x^2)dx$

3) $\int cos^2(2y) \cdot sin(2y)dy$

4) $\int sec^3 x \cdot tan\, x\ dx$

5) $\int \sqrt{2 + sin3t} \cdot cos\,3t\ dt$

6) $\int \frac{d\theta}{cos^2\theta}$

7) $\int \left(1 - sin^2 3t\right) \cdot cos3t\ dt$

8) $\int tan^3(5x) \cdot sec^2(5x)\ dx$

9) $\int sin^4 x \cdot cos^3 x\ dx$

10) $\int \frac{cot^2\sqrt{x}}{\sqrt{x}} dx$

Sol.-

1) $\frac{1}{3}\int 3\cos(3\theta - 1)d\theta = \frac{1}{3}sin(3\theta - 1) + c$

2) $\frac{1}{4}\int 4x \cdot sin(2x^2)dx = -\frac{1}{4}cos(2x^2) + c$

3) $-\frac{1}{2}\int (cos\,2y)^2 \cdot \left(-2\,sin\,2y\ dy\right) = -\frac{1}{2} \cdot \frac{(cos\,2y)^3}{3} + c = -\frac{1}{6}(cos\,2y)^3 + c$

4) $\int sec^2 x \cdot \left(sec\,x \cdot tan\,x \cdot dx\right) = \frac{sec^3 x}{3} + c$

5) $\frac{1}{3}\int (2 + sin\,3t)^{1/2}\left(3\cos 3t\ dt\right) = \frac{1}{3} \cdot \frac{(2 + sin\,3t)^{3/2}}{3/2} + c = \frac{2}{9}\sqrt{(2 + sin\,3t)^3} + c$

6) $\int \frac{d\theta}{cos^2\theta} = \int sec^2\theta \cdot d\theta = tan\,\theta + c$

7) $\int \left(1 - sin^2 3t\right) \cdot cos\,3t\ dt = \frac{1}{3}\int 3\cos 3t\ dt - \frac{1}{3}\int (sin3t)^2 \cdot 3cos3t\ dt$

$= \frac{1}{3}sin\,3t - \frac{1}{3} \cdot \frac{sin^3 3t}{3} + c = \frac{1}{3} \cdot sin\,3t - \frac{1}{9}sin^3 3t + c$

8) $\frac{1}{5}\int tan^3 5x \cdot \left(5\,sec^2 5x\ dx\right) = \frac{1}{5} \cdot \frac{tan^4 5x}{4} + c = \frac{1}{20}tan^4 5x + c$

9) $\int sin^4 x \cdot cos^3 x\ dx = \int sin^4 x \cdot \left(1 - sin^2 x\right) \cdot cos\,x\ dx$

$= \int sin^4 x \cdot cos\,x\ dx - \int sin^6 x \cdot cos\,x\ dx = \frac{sin^5 x}{5} - \frac{sin^7 x}{7} + c$

$$10)\ \int \frac{\cot^2\sqrt{x}}{\sqrt{x}}dx = \int \frac{\csc^2\sqrt{x}-1}{\sqrt{x}}dx = 2\int \frac{\csc^2\sqrt{x}}{2\sqrt{x}} - \int x^{-1/2}\,dx$$

$$= 2\left(-\cot\sqrt{x}\right) - \frac{x^{1/2}}{1/2} + c = -2\cot\sqrt{x} - 2\sqrt{x} + c$$

5-3- <u>Integrals of inverse trigonometric functions</u>:

The integration formulas for the inverse trigonometric functions are:

$$16)\ \int \frac{du}{\sqrt{a^2-u^2}} = \sin^{-1}\frac{u}{a} + c = -\cos^{-1}\frac{u}{a} + c \qquad ; \qquad \forall u^2 < a^2$$

$$17)\ \int \frac{du}{a^2+u^2} = \frac{1}{a}\tan^{-1}\frac{u}{a} + c = -\frac{1}{a}\cot^{-1}\frac{u}{a} + c$$

$$18)\ \int \frac{du}{u\sqrt{u^2-a^2}} = \frac{1}{a}\sec^{-1}\left|\frac{u}{a}\right| + c = -\frac{1}{a}\csc^{-1}\left|\frac{u}{a}\right| + c \qquad ; \qquad \forall u^2 > a^2$$

<u>EX-3</u> **Evaluate the following integrals:**

$1)\ \int \frac{x^2}{\sqrt{1-x^6}}dx$

$2)\ \int \frac{dx}{\sqrt{9-x^2}}$

$3)\ \int \frac{x}{1+x^4}dx$

$4)\ \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx$

$5)\ \int \frac{dx}{x\sqrt{4x^2-1}}$

$6)\ \int \frac{2dx}{\sqrt{x}(1+x)}$

$7)\ \int \frac{dx}{1+3x^2}$

$8)\ \int \frac{2\cos x}{1+\sin^2 x}dx$

$9)\ \int \frac{e^{\sin^{-1}x}}{\sqrt{1-x^2}}$

$10)\ \int \frac{\tan^{-1}x}{1+x^2}dx$

<u>Sol.</u>-

$$1)\ \frac{1}{3}\int \frac{1}{\sqrt{1-(x^3)^2}}\left(3x^2\,dx\right) = \frac{1}{3}\sin^{-1}x^3 + c$$

$$2)\ \int \frac{dx}{\sqrt{9-x^2}} = \sin^{-1}\frac{x}{3} + c$$

$$3)\ \frac{1}{2}\int \frac{2x}{1+(x^2)^2}dx = \frac{1}{2}\tan^{-1} x^2 + c$$

$$4)\ \int \frac{\sec^2 x}{\sqrt{1-\tan^2 x}}dx = \sin^{-1}(\tan x) + c$$

$$5)\ \int \frac{2\ dx}{2x\sqrt{(2x)^2-1}} = \sec^{-1}(2x) + c$$

$$6)\ \int \frac{2}{\sqrt{x}(1+x)}dx = 4\int \frac{\frac{1}{2\sqrt{x}}\ dx}{1+(\sqrt{x})^2} = 4\tan^{-1}\sqrt{x} + c$$

$$7)\ \frac{1}{\sqrt{3}}\int \frac{\sqrt{3}\ dx}{1+(\sqrt{3}x)^2} = \frac{1}{\sqrt{3}}\tan^{-1}(\sqrt{3}x) + c$$

$$8)\ 2\int \frac{\cos x\ dx}{1+(\sin x)^2} = 2\tan^{-1}(\sin x) + c$$

$$9)\ \int e^{\sin^{-1} x} \cdot \frac{dx}{\sqrt{1-x^2}} = e^{\sin^{-1} x} + c$$

$$10)\ \int \tan^{-1} x \cdot \frac{dx}{1+x^2} = \frac{(\tan^{-1} x)^2}{2} + c$$

5-4- Integrals of hyperbolic functions:

The integration formulas for the hyperbolic functions are:

$$19)\ \int \sinh u \cdot du = \cosh u + c$$

$$20)\ \int \cosh u \cdot du = \sinh u + c$$

$$21)\ \int \tanh u \cdot du = \ln(\cosh u) + c$$

$$22)\ \int \coth u \cdot du = \ln(\sinh u) + c$$

$$23)\ \int \operatorname{sech}^2 u \cdot du = \tanh u + c$$

$$24)\ \int \operatorname{csch}^2 u \cdot du = \coth u + c$$

$$25)\ \int \operatorname{sech} u \cdot \tanh u \cdot du = -\operatorname{sech} u + c$$

$$26)\ \int \operatorname{csch} u \cdot \coth u \cdot du = -\operatorname{csch} u + c$$

EX-4 **– Evaluate the following integrals:**

1) $\int \frac{cosh(lnx)}{x} dx$

2) $\int sinh(2x+1)\,dx$

3) $\int \frac{sinhx}{cosh^4 x}\, dx$

4) $\int x \cdot cosh(3x^2)\,dx$

5) $\int sinh^4 x \cdot cosh\,x\, dx$

6) $\int sec\,h^2(2x-3)\,dx$

7) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx$

8) $\int (e^{ax} - e^{-ax})dx$

9) $\int \frac{sinh\,x}{1+cosh\,x} dx$

10) $\int csch^2 x \cdot cothx\,dx$

Sol.-

1) $\int cosh(ln\,x)\cdot\left(\frac{dx}{x}\right) = sinh(ln\,x) + c$

2) $\frac{1}{2}\int sinh(2x+1)\cdot(2\,dx) = \frac{1}{2}cosh(2x+1)+c$

3) $\int \frac{1}{cosh^3 x}\cdot\frac{sinh\,x}{cosh\,x}\, dx = \int sec\,h^3 x \cdot tanh\,x\, dx$

$= -\int sec\,h^2 x \cdot (-sec\,hx \cdot tanh\,x\, dx) = -\frac{sec\,h^3 x}{3} + c$

4) $\frac{1}{6}\int cosh(3x^2)\cdot(6x\,dx) = \frac{1}{6}sinh(3x^2)+c$

5) $\int sinh^4 x \cdot (cosh\,x\, dx) = \frac{sinh^5 x}{5} + c$

6) $\frac{1}{2}\int sec\,h^2(2x-3)\cdot(2\, dx) = \frac{1}{2}tanh(2x-3)+c$

7) $\int \frac{e^x - e^{-x}}{e^x + e^{-x}} dx = \int tanh\,x\, dx = ln(cosh\,x) + c$

8) $2\int \frac{e^{ax} - e^{-ax}}{2} dx = \frac{2}{a}\int sinh\,ax\, (a\,dx) = \frac{2}{a}cosh\,ax + c$

9) $\int \frac{sinh\,x\, dx}{1+cosh\,x} = ln(1+cosh\,x)+c$

10) $-\int csc\,hx \cdot (-csc\,hx \cdot coth\,x\, dx) = -\frac{csc\,h^2 x}{2} + c$

5-5- <u>Integrals of inverse hyperbolic functions</u>:

The integration formulas for the inverse hyperbolic functions are:

$$27)\ \int \frac{du}{\sqrt{1+u^2}} = \sinh^{-1} u + c$$

$$28)\ \int \frac{du}{\sqrt{u^2-1}} = \cosh^{-1} u + c$$

$$29)\ \int \frac{du}{1-u^2} = \begin{cases} \tanh^{-1} u + c & \text{if } |u|<1 \\ \coth^{-1} u + c & \text{if } |u|>1 \end{cases} = \frac{1}{2}\ln\left|\frac{1+u}{1-u}\right| + c$$

$$30)\ \int \frac{du}{u\sqrt{1-u^2}} = -\operatorname{sech}^{-1}|u| + c = -\cosh^{-1}\left(\frac{1}{|u|}\right) + c$$

$$31)\ \int \frac{du}{u\sqrt{1+u^2}} = -\operatorname{csch}^{-1}|u| + c = -\sinh^{-1}\left(\frac{1}{|u|}\right) + c$$

***<u>EX-4</u>* – Evaluate the following integrals:**

$$1)\ \int \frac{dx}{\sqrt{1+4x^2}} \qquad 2)\ \int \frac{dx}{\sqrt{4+x^2}} \qquad 3)\ \int \frac{dx}{1-x^2}$$

$$4)\ \int \frac{dx}{x\sqrt{4+x^2}} \qquad 5)\ \int \frac{\sec^2\theta\ d\theta}{\sqrt{\tan^2\theta-1}} \qquad 6)\ \int \tanh^{-1}\left(\ln\sqrt{x}\right)\cdot\frac{dx}{x\left(1-\ln^2\sqrt{x}\right)}$$

***<u>Sol.</u>*-**

$$1)\ \frac{1}{2}\int \frac{2\ dx}{\sqrt{1+4x^2}} = \frac{1}{2}\sinh^{-1} 2x + c$$

$$2)\ \int \frac{\frac{1}{2}\ dx}{\sqrt{1+\left(\frac{x}{2}\right)^2}} = \sinh^{-1}\frac{x}{2} + c$$

$$3)\ \int \frac{dx}{1-x^2} = \tanh^{-1} x + c \qquad \text{if} \quad |x|<1$$

$$= \coth^{-1} x + c \qquad \text{if} \quad |x|>1$$

$$4)\ \int \frac{dx}{x\sqrt{4+x^2}} = \frac{1}{2}\int \frac{\frac{1}{2}\ dx}{\frac{x}{2}\sqrt{1+\left(\frac{x}{2}\right)^2}} = -\frac{1}{2}\operatorname{csc}h^{-1}\left|\frac{x}{2}\right| + c$$

$$5)\ \int \frac{1}{\sqrt{\tan^2\theta - 1}}\left(\sec^2\theta\ \ d\theta\right) = \cosh^{-1}(\tan\theta) + c$$

$$6)\quad let\quad u = \ln\sqrt{x} = \frac{1}{2}\ln x \qquad\qquad du = \frac{1}{2x}dx$$

$$\int \tanh^{-1}(\ln\sqrt{x})\cdot\frac{dx}{x(1-\ln^2\sqrt{x})} = \int \tanh^{-1} u\cdot\frac{2\,du}{1-u^2}$$

$$= 2\frac{\left(\tanh^{-1} u\right)^2}{2} + c = \left[\tanh^{-1}(\ln\sqrt{x})\right]^2 + c$$

Problems – 5

Evaluate the following integrals:

1) $\int (x^2 - 1)\cdot(4 - x^2)dx$ (ans.: $\frac{5}{3}x^3 - \frac{1}{5}x^5 - 4x + c$)

2) $\int e^x \cdot \sin e^x \, dx$ (ans.: $-\cos e^x + c$)

3) $\int \tan(3x+5)\, dx$ (ans.: $-\frac{1}{3}\ln|\cos(3x+5)| + c$)

4) $\int \frac{\cot(\ln x)}{x} dx$ (ans.: $\ln|\sin(\ln x)| + c$)

5) $\int \frac{\sin x + \cos x}{\cos x} dx$ (ans.: $-\ln|\cos x| + x + c$)

6) $\int \frac{dx}{1+\cos x}$ (ans.: $-\cot x + \csc x + c$)

7) $\int \cot(2x+1)\cdot \csc^2(2x+1)\, dx$ (ans.: $-\frac{1}{4}\cot^2(2x+1) + c$)

8) $\int \frac{dx}{\sqrt{1-9x^2}}$ (ans.: $\frac{1}{3}\sin^{-1}(3x) + c$)

9) $\int \frac{dx}{\sqrt{2-x^2}}$ (ans.: $\sin^{-1}\frac{x}{\sqrt{2}} + c$)

10) $\int e^{2x} \cdot \cosh e^{2x}\, dx$ (ans.: $\frac{1}{2}\sinh e^{2x} + c$)

11) $\int e^{\sin x} \cdot \cos x \, dx$ (ans.: $e^{\sin x} + c$)

12) $\int \frac{dx}{e^{3x}}$ (ans.: $-\frac{1}{3}e^{-3x} + c$)

13) $\int \frac{e^{\sqrt{x}} - 1}{\sqrt{x}} dx$ (ans.: $2e^{\sqrt{x}} - 2\sqrt{x} + c$)

14) $\int x(a + b\sqrt{3x})dx$ where a, b constants (ans.: $\frac{1}{10}(5ax^2 + 4\sqrt{3}bx^{5/2}) + c$)

15) $\int \frac{dx}{-1-x^2}$ (ans.: $-\tan^{-1}x + c$)

16) $\int \frac{\cos\theta \, d\theta}{1+\sin^2\theta}$ (ans.: $\tan^{-1}(\sin\theta) + c$)

17) $\int \frac{1}{x^2}\csc\frac{1}{x}\cot\frac{1}{x}\,dx$ $\quad$ $(ans.:\ \csc\frac{1}{x}+c)$

18) $\int \frac{3x+1}{\sqrt[3]{3x^2+2x+1}}\,dx$ $\quad$ $(ans.:\ \frac{3}{4}\sqrt[3]{(3x^2+2x+1)^2}+c)$

19) $\int \sin(\tan\theta)\cdot\sec^2\theta\,d\theta$ $\quad$ $(ans.:\ -\cos(\tan\theta)+c)$

20) $\int \sqrt{x^2-x^4}\,dx$ $\quad$ $(ans.:\ -\frac{1}{3}\sqrt{(1-x^2)^3}+c)$

21) $\int \frac{\sec^2 2x\ dx}{\sqrt{\tan 2x}}$ $\quad$ $(ans.:\ \sqrt{\tan 2x}+c)$

22) $\int (\sin\theta-\cos\theta)^2\,d\theta$ $\quad$ $(ans.:\ \theta+\cos^2\theta+c)$

23) $\int \frac{y}{y^4+1}\,dy$ $\quad$ $(ans.:\ \frac{1}{2}\tan^{-1}y^2+c)$

24) $\int \frac{dx}{\sqrt{x}(x+1)}$ $\quad$ $(ans.:\ 2\tan^{-1}\sqrt{x}+c)$

25) $\int t^{\frac{2}{3}}(t^{\frac{5}{3}}+1)^{\frac{2}{3}}\,dt$ $\quad$ $(ans.:\ \frac{9}{25}(t^{\frac{5}{3}}+1)^{\frac{5}{3}}+c)$

26) $\int \frac{dx}{x^{\frac{1}{5}}\sqrt{1+x^{\frac{4}{5}}}}$ $\quad$ $(ans.:\ \frac{5}{2}\sqrt{1+x^{\frac{4}{5}}}+c)$

27) $\int \frac{(\cos^{-1}4x)^2}{\sqrt{1-16x^2}}\,dx$ $\quad$ $(ans.:\ -\frac{1}{12}(\cos^{-1}4x)^3+c)$

28) $\int \frac{dx}{x\sqrt{4x^2-1}}$ $\quad$ $(ans.:\ \sec^{-1}(2x)+c)$

29) $\int \frac{dx}{(e^x+e^{-x})^2}$ $\quad$ $(ans.:\ \frac{1}{4}\tanh x+c)$

30) $\int 3^{\ln x^2}\frac{dx}{x}$ $\quad$ $(ans.:\ \frac{1}{2\ln 3}3^{\ln x^2}+c)$

31) $\int \frac{\cot x\ dx}{\ln(\sin x)}$ $\quad$ $(ans.:\ \ln\ln(\sin x)+c)$

32) $\int \frac{(\ln x)^2}{x}\,dx$ $\quad$ $(ans.:\ \frac{1}{3}(\ln x)^3+c)$

33) $\int \frac{\sin x\cdot e^{\sec x}}{\cos^2 x}\,dx$ $\quad$ $(ans.:\ e^{\sec x}+c)$

$34)\ \int \frac{dx}{x \cdot \ln x}$ $(ans.:\ \ln \ln x + c)$

$35)\ \int \frac{d\theta}{\cosh\theta + \sinh\theta}$ $(ans.:\ -e^{-\theta} + c)$

$36)\ \int \frac{2^x - 8^{2x}}{\sqrt{4^x}}\,dx$ $(ans.:\ x - \frac{1}{5\ln 2}2^{5x} + c)$

$37)\ \int \frac{e^{\tan^{-1}2t}}{1+4t^2}\,dt$ $(ans.:\ \frac{1}{2}e^{\tan^{-1}2t} + c)$

$38)\ \int \frac{\cot x}{\csc x}\,dx$ $(ans.:\ \sin x + c)$

$39)\ \int \sec^4 x \cdot \tan^3 x\ dx$ $(ans.:\ \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + c)$

$40)\ \int \csc^4 3x\ dx$ $(ans.:\ -\frac{1}{9}\cot^3 3x - \frac{1}{3}\cot 3x + c)$

$41)\ \int \frac{\cos^3 t}{\sin^2 t}\,dt$ $(ans.:\ -\csc t - \sin t + c)$

$42)\ \int \frac{\sec^4 x}{\tan^4 x}\,dx$ $(ans.:\ -\frac{1}{3}\cot^3 x - \cot x + c)$

$43)\ \int \tan^2 4\theta\ d\theta$ $(ans.:\ \frac{1}{4}\tan 4\theta - \theta + c)$

$44)\ \int \frac{e^x}{1+e^x}\,dx$ $(ans.:\ \ln(1+e^x) + c)$

$45)\ \int \tan^3 2x\ dx$ $(ans.:\ \frac{1}{4}\tan^2 2x + \frac{1}{2}\ln|\cos 2x| + c)$

$46)\ \int \frac{\sec^2 x}{2+\tan x}\,dx$ $(ans.:\ \ln(2+\tan x) + c)$

$47)\ \int \sec^4 3x\ dx$ $(ans.:\ \frac{1}{9}\tan^3 3x + \frac{1}{3}\tan 3x + c)$

$48)\ \int \frac{e^t}{1+e^{2t}}\,dt$ $(ans.:\ \tan^{-1}e^t + c)$

$49)\ \int \frac{\cos\sqrt{x}}{\sqrt{x}}\,dx$ $(ans.:\ 2\sin\sqrt{x} + c)$

$50)\ \int \frac{dx}{\sin x \cdot \cos x}$ $(ans.:\ -\ln|\csc 2x + \cot 2x| + c)$

51) $\int \sqrt{1+\sin y}\ dy$ $\qquad$ (ans.: $-2\sqrt{1-\sin y}+c$)

52) $\int \frac{dx}{(x^2+1)(2+\tan^{-1} x)}$ $\qquad$ (ans.: $\ln(2+\tan^{-1} x)+c$)

53) $\int \sin^{-1}(\cosh x)\cdot \frac{\sinh x\ dx}{\sqrt{1-\cosh^2 x}}$ $\qquad$ (ans.: $\frac{1}{2}\left(\sinh^{-1}(\cosh x)\right)^2+c$)

54) $\int \frac{\cos\theta\ d\theta}{1-\sin^2\theta}$ $\qquad$ (ans.: $\ln\left|\sec\theta+\tan\theta\right|+c$)

55) $\int \frac{dx}{x\left(1+(\ln x)^2\right)}$ $\qquad$ (ans.: $\tan^{-1}(\ln x)+c$)

56) $\int \left(e^{\frac{9}{4}x}-2e^{\frac{5}{4}x}+e^{\frac{x}{4}}\right) dx$ $\qquad$ (ans.: $\frac{4}{9}e^{\frac{9}{4}x}-\frac{8}{5}e^{\frac{5}{4}x}+4e^{\frac{x}{4}}+c$)

57) $\int \frac{e^x\, dx}{e^{2x}+2e^x+1}$ $\qquad$ (ans.: $-\frac{1}{e^x+1}+c$)

58) $\int e^x\cdot \sinh 2x\ dx$ $\qquad$ (ans.: $\frac{1}{2}\left[\frac{1}{3}e^{3x}+e^{-x}\right]+c$)

59) $\int \frac{\sec^3 x+e^{\sin x}}{\sec x}\, dx$ $\qquad$ (ans.: $\tan x+e^{\sin x}+c$)

60) $\int \frac{3^{x+2}}{2+9^{x+1}}\, dx$ $\qquad$ (ans.: $\frac{3}{\sqrt{2}\ln 3}\tan^{-1}\frac{3^{x+1}}{\sqrt{2}}+c$)

61) $\int \frac{\cos x\ dx}{\sqrt{\sin x}\cdot\sqrt{1-\sin x}}$ $\qquad$ (ans.: $2\sin^{-1}\sqrt{\sin x}+c$)

62) $\int \tan^5 x\ dx$ $\qquad$ (ans.: $\frac{1}{4}\sec^4 x-\sec^2 x-\ln\left|\cos x\right|+c$)

63) $\int e^{\ln \sin^{-1} x}\cdot \frac{dx}{\sqrt{1-x^2}}$ $\qquad$ (ans.: $\frac{1}{2}(\sin^{-1} x)^2+c$)

64) $\int x\cdot e^{x^2-1}\ dx$ $\qquad$ (ans.: $\frac{1}{2}e^{x^2-1}+c$)

65) $\int \cosh(\ln \cos x)\ dx$ $\qquad$ (ans.: $\frac{1}{2}\left[\sin x+\ln\left|\sec x+\tan x\right|\right]+c$)

66) $\int \frac{\cos x}{\sin^2 x}\, dx$ $\qquad$ (ans.: $-\csc x+c$)

67) $\int \cosh^{-1}(\sin x)\ \frac{\cos x\, dx}{\sqrt{\sin^2 x-1}}$ $\qquad$ (ans.: $\frac{1}{2}\left[\cosh^{-1}(\sin x)\right]^2+c$)