عنوان المحاضرة:

Vectors, Vectors in Space, Unit Vector, Scalar Product, Vector Product, line and plane equations المتجهات، المتجهات في الفضاء، وحدة المتجه، ضرب القيمة العددية، ضرب المتجه، معادلة الخط والمستوى

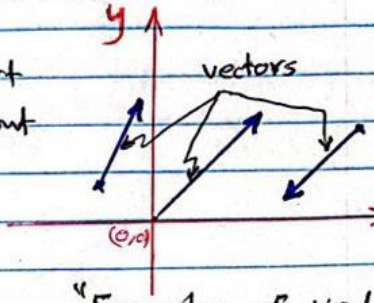
Vectors :- المتجهات

- The vector is only two pieces of information:
1- Direction
2- Length or Magnitude

المتجه عبارة عن مكونين وهما : الاتجاه والطول أو الكمية.

- We can graph a vector by an arrow that we can visualize on x-y plane and we can capture it by the arrow length and angle

- Vectors on graph could start from not just an origion, but from any where.



"Examples of vectors"

Examples of vectors :-

- To answer the question "What is the current temperature?" we use a single number (scalar) ; likewise the question about a mass ;
- While to answer the question "What is the current velocity of the wind?", we need more than just a single number. We need magnitude (speed) and direction. This where vectors come to handy.

positions displacement, velocity, acceleration, force, momentum & torque are all physical quantities that can be represented mathmatically by vectors.

Vector Denoting :-

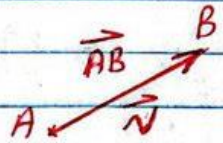
- Vectors are writing with an arrow on top on equations.

Ex)

- Velocity vector ⟶ $\vec{v}$
- Force vector ⟶ $\vec{F}$

- Note) Any varrable symbol with no arrow on top means scalar.

- A vector can be geometrically represented by a directed line segment with a head & a tail ;



So vector $\vec{AB}$ is a vector from point A to B.

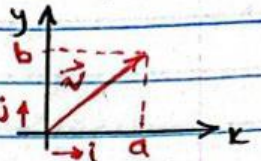
- Also, we can denote vector $\vec{AB}$ by a small case letter $\vec{v}$
- The length of the arrow corresponds to the magnitude of the vector.
- The arrow points in the direction of the vector.

How to represent the vector mathematically :-

Vector in plane :-

We can write vectors as columns. Let us take a very important special vector as example :-

$$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad , \quad j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

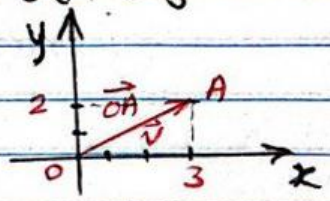


Any vector in the xy-plane can be written in terms of i & j using the triangle law & scalar multiplication.

$$\vec{v} = ai + bj = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

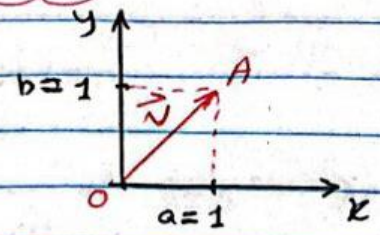
Ex/

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3i + 2j$$



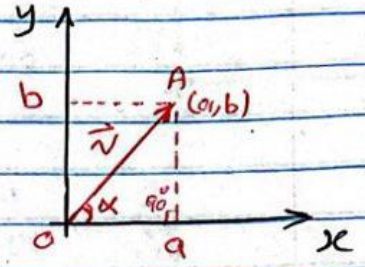
Note/

* IF $a = b = 1$, then $\vec{v} = i + j$ is a "unit vector"



Finding the length/magnitude and the direction of vector ie

If $\vec{v} = a\,i + b\,j$ ---①, then the length/magnitude of vector $\vec{v}$ is :-



$$|v| = \sqrt{a^2 + b^2} \quad \text{---②}$$

- It's a Pythagorean theorem

$$\left.\begin{aligned} a &= |v| \cos\alpha \\ b &= |v| \sin\alpha \end{aligned}\right\} \text{---③}$$

$$\tan\alpha = \frac{b}{a}$$

Substitute eq:- ③ in ① yields ;

$$\vec{v} = |v|(\cos\alpha\, i + \sin\alpha\, j)$$

$\vec{v}$ — vector symbol
$|v|$ — vector length
i, j — unit vector components (basis/Fundamental vector components)
α — vector angle with x-axis

Ex) Find a vector in plane of length (7 units) & makes angle (35°) with x-axis ?

Solution

since $|\vec{v}| = 7$ & $\alpha = 35^\circ$

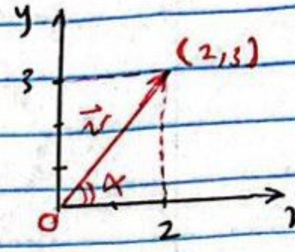
$\therefore \vec{v} = 7(\cos 35\, i + \sin 35\, j)$

$\vec{v} = 5.7\, i + 4\, j$ Ans

Scanned with CamScanner

Ex) Find the angle between the vector $\vec{v} = 2i + 3j$ and the x-axis?

Solution



$|\vec{v}| = \sqrt{a^2 + b^2}$; $a = 2$, $b = 3$

$|\vec{v}| = \sqrt{2^2 + 3^2} = \sqrt{13}$

$a = |v| \cos\alpha \Longrightarrow \cos\alpha = \frac{a}{|v|} = \frac{2}{\sqrt{13}}$

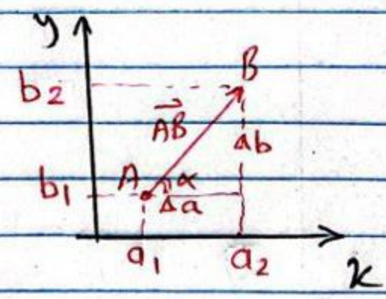
$\therefore \alpha = \cos^{-1} \frac{2}{\sqrt{13}} = \boxed{56.3^\circ}$ Ans

---

Vectors with tail not in origion :-

المتجهات يبتدئ نقطة ليس ع (0,0)

Vectors can be start not from the origion, but from any where like A to B

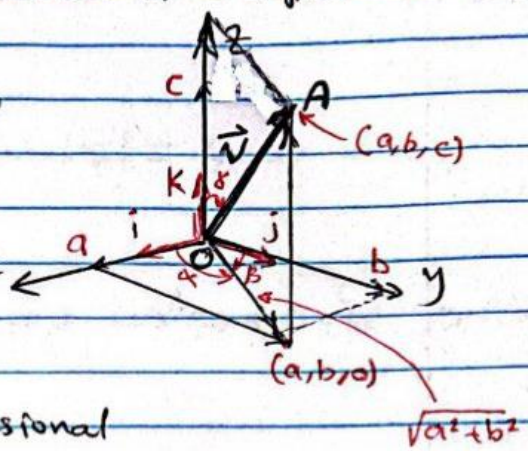


$\therefore \vec{AB} = \Delta a\, i + \Delta b\, j$

$\vec{AB} = (a_2 - a_1)\, i + (b_2 - b_1)\, j$

Vectors in a space :-

- This could be in three or higher dimensions
- Similar to the 2-D case, but we now have three basis vectors i, j & k.
- From these three components unit vectors we can describe any vector in three-dimensional space.



$$\vec{v} = \vec{OA} = a\,i + b\,j + c\,K$$

where in

i, j, K = Basis or Fundamental unit vector.

a, b, c = Directional numbers (scalars).

α, β, γ = Directional angles.

$$|v| = |OA| = \sqrt{a^2+b^2+c^2}$$

$$a = |v| \times \cos\alpha$$
$$b = |v| \times \cos\beta$$
$$c = |v| \times \cos\gamma$$

$$\frac{\vec{v}}{|v|} = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, K$$ = unit vector with the direction of $\vec{v}$

And,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Ex) Find a vector in space of length (5 units) that makes angles ($70°$) with x-axis, ($85°$) with y-axis ?

Solution:

$\alpha = 70°$, $\beta = 85°$, $|\vec{v}| = 5$

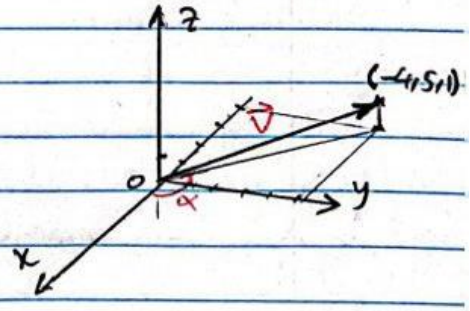
$\gamma = ?$, $\vec{v} = ?$

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \Rightarrow \cos^2 70 + \cos^2 85 + \cos^2\gamma = 1$$

$$\therefore \boxed{\cos\gamma = 0.935}$$

$$\therefore \vec{v} = |\vec{v}|(\cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k)$$
$$= 5(\cos 70\, i + \cos 85\, j + 0.935\, k)$$
$$\boxed{\vec{v} = 1.7\, i + 0.436\, j + 4.675\, k}$$ Ans

Ex) Find the angle between the vector $\vec{v} = -4i + 5j + k$ and the x-axis?



Solution:

$a = -4$, $b = 5$, $c = 1$

$$|\vec{v}| = \sqrt{a^2 + b^2 + c^2}$$

$$|\vec{v}| = \sqrt{(-4)^2 + (5)^2 + (1)^2} = \sqrt{42}$$

$$\cos\alpha = \frac{a}{|\vec{v}|} \Rightarrow \alpha = \cos^{-1}\frac{a}{|\vec{v}|} = \cos^{-1}\frac{-4}{\sqrt{42}}$$

$$\boxed{\alpha = 128°}$$

Scalar product :- (Dot product)
ضرب القيم العددية

Let $\vec{A} = a_1 i + a_2 j + a_3 k$
And $\vec{B} = b_1 i + b_2 j + b_3 k$

Then, $\vec{A} \cdot \vec{B} = |A||B| \cos\theta$

Where θ — is the angle between $\vec{A}$ & $\vec{B}$

Properties :-

1- $\vec{A} \cdot \vec{A} = |A|^2$

2- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} = a_1 b_1 + a_2 b_2 + a_3 b_3$

3- $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|A||B|} = \dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} + \sqrt{b_1^2 + b_2^2 + b_3^2}}$

4- $\vec{A} \perp \vec{B} \Rightarrow \vec{A} \cdot \vec{B} = 0$ [Orthogonal Vectors] متجهات متعامدة

5- $ai + bj \perp bi - aj$

Ex) Find the angle "θ" between $\vec{A} = i - 2j - 2k$ & $\vec{B} = 6i + 3j + 2k$?

Solution

$\vec{A} \cdot \vec{B} = (1 * 6) + (-2 * 3) + (-2 * 2) = \boxed{-4}$

$|A| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = \boxed{3}$

$|B| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = \boxed{7}$

$\Rightarrow |A||B| = \boxed{21}$

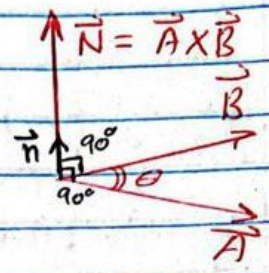
$\therefore \cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|A||B|} \Rightarrow \theta = \cos^{-1}\dfrac{-4}{21} \approx \boxed{101^\circ}$ Ans

Scanned with CamScanner

Vector Product :- (Cross-product)
ضرب المتجهات

Normal vector is what yields from vector product or cross product.

$$\vec{N} = \vec{A} \times \vec{B} = \vec{n}\,|A||B| \sin\theta$$



where $\vec{n}$ is a normal unit vector

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \text{ ; where, } \vec{A} = a_1 i + a_2 j + a_3 k, \quad \vec{B} = b_1 i + b_2 j + b_3 k$$

Properties :-

1- $\vec{A} \times \vec{A} = 0 \longrightarrow$ "$\sin 0 = 0$"

2- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

3- $\vec{A} \parallel \vec{B} \Longrightarrow \vec{A} \times \vec{B} = 0 \longrightarrow \sin 0 = 0$

4- Area of $\Delta ABC = \frac{1}{2} |\vec{A} \times \vec{B}|$

Ex) Find $\vec{A}\times\vec{B}$ & $\vec{B}\times\vec{A}$ if

$\vec{A} = 2i+j+k$

$\vec{B} = -4i+3j+k$

Solution:

$$\vec{A}\times\vec{B} = \begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i - \begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j + \begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$

$$= (1\times1 - (3\times1))\,i - (2\times1 - (-4\times1))\,j + (2\times3 - (-4\times1))\,k$$

$$\boxed{\vec{A}\times\vec{B} = -2i - 6j + 10k}$$

but,

$$\vec{B}\times\vec{A} = -\vec{A}\times\vec{B} = \boxed{2i+6j-10k}$$

Triple product :- الضرب الثلاثي لمتجهات

A- Scalar triple product :-

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = (\vec{A}\times\vec{B})\cdot\vec{C}$$

Note)

1- Box volume is $\Longrightarrow$ $V_{box} = |\vec{A}\cdot\vec{B}\times\vec{C}|$

2- Pyramid Volume is $\Longrightarrow$ $V_{Py.} = \frac{1}{6}|\vec{A}\cdot\vec{B}\times\vec{C}|$

B- Vector triple Product :-

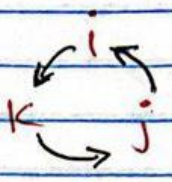
$$\vec{A}\times(\vec{B}\times\vec{C}) = (\vec{A}\cdot\vec{C})\times B - (\vec{A}\cdot\vec{B})\,\vec{C}$$

Note)

$i\cdot i = j\cdot j = k\cdot k = 1$; $i\times i = j\times j = k\times k = 0$

Scanned with CamScanner

$i \cdot j = j \cdot k = k \cdot i = 0$



$i \times j = k$
$j \times k = i$
$k \times i = j$

HW#2

1- Find the length & direction of these vectors & the angles make with the x-axis?
   a- $5i + 12j$
   b- $\sqrt{3}i + j$

2- Find a vector 6 units long in the direction of $\vec{A} = 2i + 2j - k$

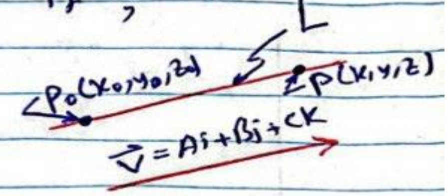
3- Find the area of the triangle whose vertices are $A(1,-1,0)$, $B(2,1,-1)$, $C(-1,1,2)$?

4- If $\vec{A} = 2i - j$ & $B = i + 3j - 2k$, Find $A \times B$, then calculate $(A \times B) \cdot A$?

Equation of line in a space معادلة الخط المستقيم في الفضاء

If L is a line in a space that passes through a point $P_0(x_0, y_0, z_0)$ & it parallel to a vector $\vec{V} = Ai + Bj + Ck$, Then $P(x, y, z)$ is any point lies on L only if ;

$$\overrightarrow{P_0P} = t\vec{V} \quad \text{——} \quad (1)$$



Where ;
t — time parameter

So eq. (1) can be written as ;

$$(x - x_0)i + (y - y_0)j + (z - z_0)k = t(Ai + Bj + Ck)$$

$x - x_0 = At \Longrightarrow x = At + x_0$
$y - y_0 = Bt \Longrightarrow y = Bt + y_0$
$z - z_0 = Ct \Longrightarrow z = Ct + z_0$

← Eq. of line in a space

Ex1

Find parametric equations for the line through the point $(-2, 0, 4)$ parallel to the vector $\vec{V} = 2i + 4j - 2k$

Solution

$P_0(x_0, y_0, z_0) = (-2, 0, 4)$

$Ai + Bj + Ck = 2i + 4j - 2k$

$\therefore \boxed{x = 2t - 2}$

$\boxed{y = 4t}$; $\boxed{z = -2t + 4}$

Scanned with CamScanner

Ex)

Find parametric equations for line through the points $P(-3,2,-3)$ & $Q(1,-1,4)$?

(Solution)

$$\vec{PQ} = (1-(-3))\,i + (-1-2)\,j + (4-(-3))\,k$$

$$\boxed{\vec{PQ} = 4i - 3j + 7k} = \boxed{Ai + Bj + Ck}$$

* Let $(x_0, y_0, z_0) = (-3, 2, -3)$

$$\therefore \boxed{x = 4t - 3} \; ; \; \boxed{y = -3t + 2} \; ; \; \boxed{z = 7t - 3}$$

or;

* Let $(x_0, y_0, z_0) = (1, -1, 4)$

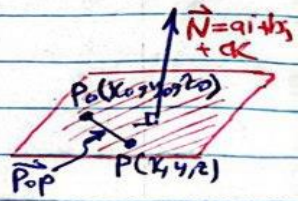
$$\therefore \boxed{x = 4t + 1} \; ; \; \boxed{y = -3t + 1} \; ; \; \boxed{z = 7t + 4}$$

Equation of the plane : معادلة المستوي

To Find the eqn of the plane that passes through the point $P_0(x_0, y_0, z_0)$ & it's normal Vector is $\vec{N} = ai + bj + ck$

Let $P(x,y,z)$ be any point in the plane

$$\vec{P_0P} = (x-x_0)i + (y-y_0)j + (z-z_0)k$$



$$\vec{P_0P} \perp N \Longrightarrow \vec{P_0P} \cdot \vec{N} = 0$$

$$\Longrightarrow a(x-x_0) + b(y-y_0) + c(z-z_0) = 0$$

$$\Longrightarrow ax + by + cz = ax_0 + by_0 + cz_0$$

or

$$\boxed{ax + by + cz = d}$$ ← Equation of the plane

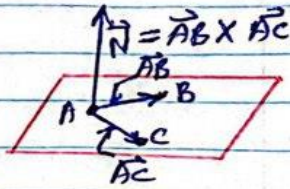
Scanned with CamScanner

Ex1

Find the equation of the plane having the points $A(2,3,5)$, $B(7,2,1)$ & $C(1,1,1)$

Solution

في هذا المثال نوجد نقاط ثلاثة من هذه النقاط الثلاثة يمكننا ايجاد متجهين اثنين، وليكونا $\vec{AB}$ & $\vec{AC}$



ولكوننا نحتاج $\vec{N}$ فهنا نستطيع ايجاد Cross prod. او vector prod. لكل من $\vec{N}$ بين $\vec{AC}$ و $\vec{AB}$ وكما يأتي:-

$$\vec{AB} = (7-2)i + (2-3)j + (1-5)k$$
$$\boxed{\vec{AB} = 5i - j - 4k}$$

$$\vec{AC} = (1-2)i + (1-3)j + (1-5)k$$
$$\boxed{\vec{AC} = -i - 2j - 4k}$$

$$\therefore \vec{N} = \vec{AB} \times \vec{AC} = \begin{vmatrix} i & j & k \\ 5 & -1 & -4 \\ -1 & -2 & -4 \end{vmatrix} = i\begin{vmatrix} -1 & -4 \\ -2 & -4 \end{vmatrix} - j\begin{vmatrix} 5 & -4 \\ -1 & -4 \end{vmatrix} + k\begin{vmatrix} 5 & -1 \\ -1 & -2 \end{vmatrix}$$

$$\vec{N} = \boxed{-4i + 24j - 11k}$$

∴ To find the value of "d", we do a substitution of any point A, B, or C as follows;

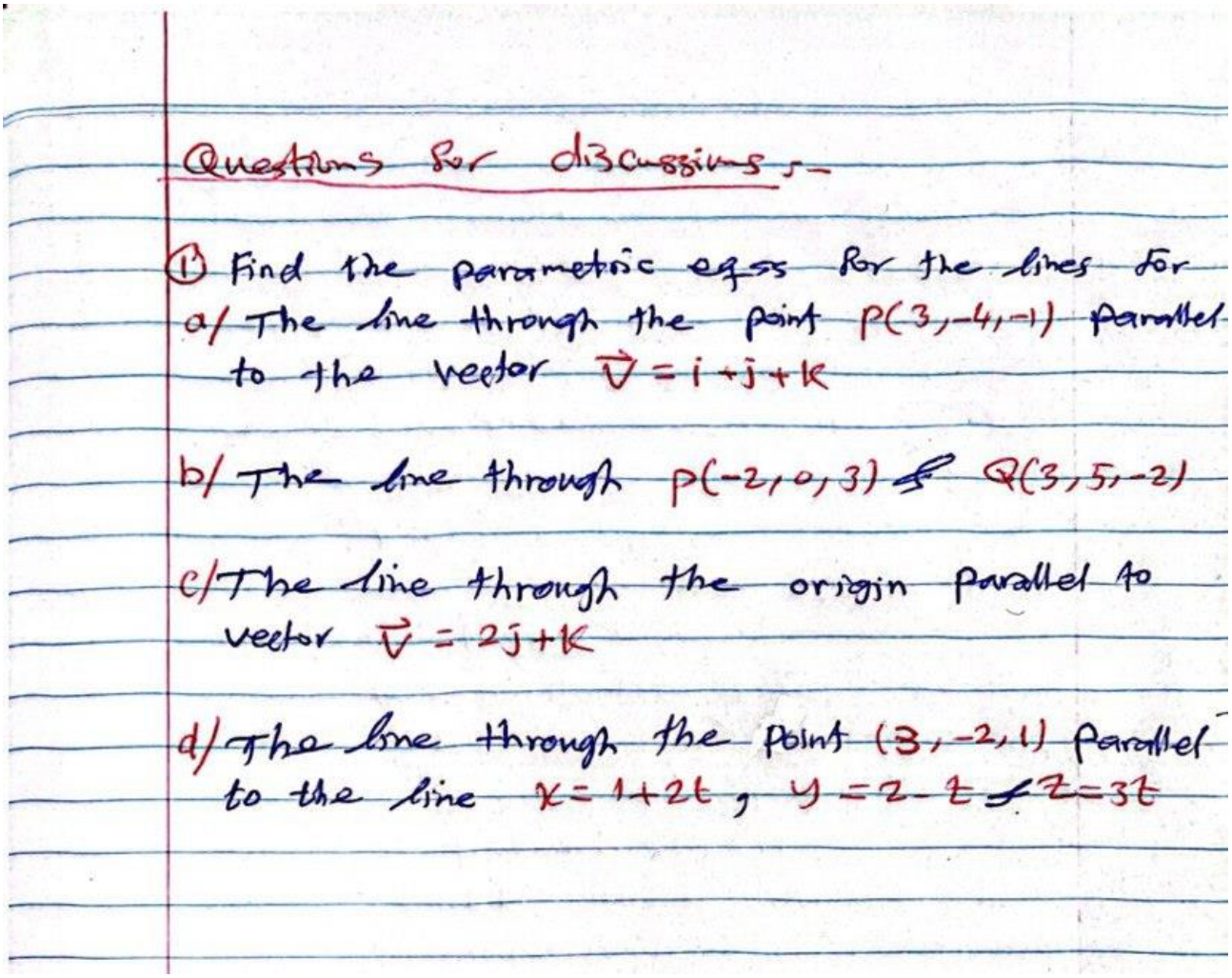
$ax + by + cz = d$; $d = ax_0 + by_0 + cz_0$

let $P_0 \Rightarrow C(1,1,1) \Rightarrow d = (-4*1) + (24*1) + (-11*1)$

$$\boxed{d = -9}$$

$$\therefore \boxed{-4x + 24y - 11z = -9} \leftarrow \text{Eq. of the plane}$$

Scanned with CamScanner



Questions for discussions :-

1) Find the parametric eqns for the lines for
a/ The line through the point $P(3,-4,-1)$ parallel to the vector $\vec{v} = i+j+k$

b/ The line through $P(-2,0,3)$ & $Q(3,5,-2)$

c/ The line through the origin parallel to vector $\vec{v} = 2j+k$

d/ The line through the point $(3,-2,1)$ parallel to the line $x = 1+2t$, $y = 2-t$ & $z = 3t$

-- نهاية محاضرة " Vectors, Vectors in Space, Unit Vector, Scalar Product, Vector Product, line and plane equations المتجهات، المتجهات في الفضاء، وحدة المتجه، ضرب القيمة العددية، ضرب المتجه، معادلة الخط والمستوى"--