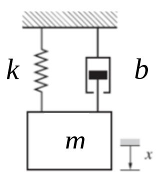
Mathematical background; Laplace transform, complex variable, matrices.

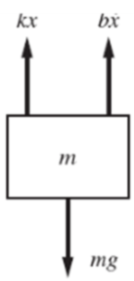
Mathematical Modelling

A mathematical model is an abstract description of a concrete system using mathematical concepts. Mathematical models may assume many different forms. Depending on the particular system and the particular circumstances, one mathematical model may be better suited than other models. For example, in optimal control problems, it is advantageous to use state-space representations. On the other hand, for the transient-response or frequency-response analysis of single input, single-output, linear, time-invariant systems, the transfer-function representation may be more convenient than any other. Once a mathematical model of a system is obtained, various analytical and computer tools can be used for analysis and synthesis purposes.

1. Mechanical Systems

Derive the mathematical model of the mechanical system.
- Displacement is positive downward, measured from the **undeflected** position of the spring





Free-body diagram

Using the free-body diagram (above and right) and summing forces we obtain the mathematical model

$$+\downarrow \sum F = -kx - b\dot{x} + mg = m\ddot{x}$$

$$\boxed{m\ddot{x} + b\dot{x} + kx = mg}$$

Model
(2^{nd}-order ODE)

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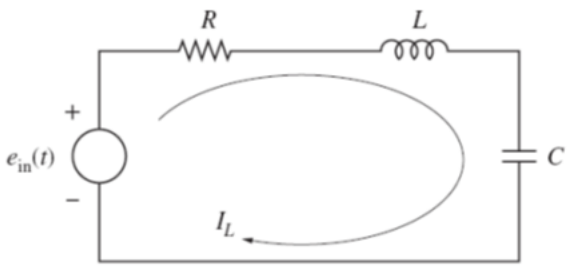
2. Electrical Systems

- Derive the mathematical model of the simple RLC circuit



Start with 1st-order ODEs for the energy-storage elements:

$C\dot{e}_C = I_L$ Capacitor

$L\dot{I}_L = e_L$ Inductor

- Now use KVL to find expression for inductor voltage e_L

$-e_R - e_L - e_C + e_{in}(t) = 0$ Clockwise KVL

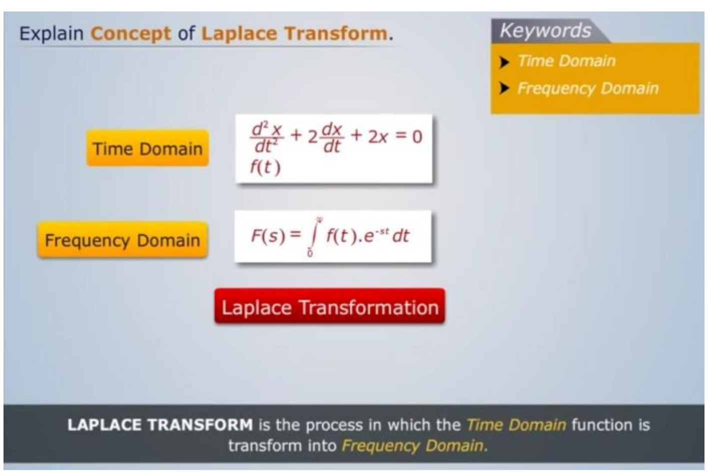
Sub KVL into inductor ODE (use Ohm's law for e_R)

$$C\dot{e}_C - I_L = 0$$
$$L\dot{I}_L + RI_L + e_C = e_{in}(t)$$

Model

2 energy storage elements ➔ 2nd-order model

Laplace Transformation



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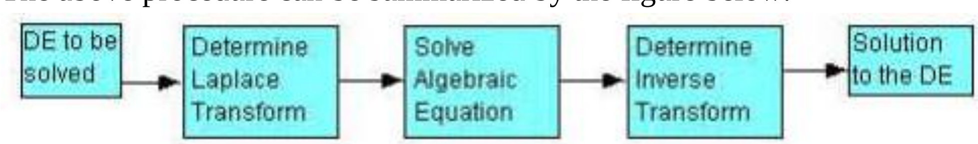
The Laplace Transform: Basic Definitions and Results

- The given \hard" problem is transformed into a \simple" equation.
- This simple equation is solved by purely algebraic manipulations.
- The solution of the simple equation is transformed back to obtain the solution of the given problem.
- In this way the Laplace transformation reduces the problem of solving a differential equation to an algebraic problem. The third step is made easier by tables, whose role is similar to that of integral tables in integration.

The above procedure can be summarized by the figure below:



Definition: The Laplace transform is defined by the linear transformation:

$$L\{f(t)\} = F(s) = \int_0^{\infty} f(t)e^{-st}dt \qquad (4.3)$$

where $s = \sigma + j\omega$ is an arbitrary complex number and f(t) is of exponential order (i.e. $f(t)e^{-st}$ is bounded for large t).

Some Laplace Transform Pairs $[f(t) \Leftrightarrow F(s)]$

Unit Impulse

$$f(t) = \delta(t) \qquad \text{and} \qquad F(s) = 1 \qquad (4.5)$$

where

$$F(s) = \int_0^{\infty} \delta(t)e^{-st}dt = 1$$

Unit Step:

$$f(t)=u(t)=\begin{cases}0 & t<0\\ 1 & t>0\end{cases} \qquad \text{and} \qquad F(s)=\frac{1}{s}$$

where

$$F(s)=\int_0^{\infty}(1)e^{-st}dt=\frac{-1}{s}e^{-st}\Big|_0^{\infty}=\frac{-1}{s}(0-1)=\frac{1}{s}$$

Exponential:

$$f(t)=e^{-at} \qquad \text{and} \qquad F(s)=\frac{1}{s+a} \qquad (4.7)$$

where

$$F(s)=\int_0^{\infty}e^{-at}e^{-st}dt=\int_0^{\infty}e^{-(s+a)t}dt=\frac{-1}{s+a}e^{-(s+a)t}\Big|_0^{\infty}=\frac{1}{s+a}$$

Ramp:

$$f(t)=at \qquad \text{and} \qquad F(s)=\frac{a}{s^2} \qquad (4.8)$$

where

$$F(s)=\int_0^{\infty}ate^{-st}dt=a\left[t\frac{e^{-st}}{-s}\Big|_0^{\infty}-\int_0^{\infty}\frac{e^{-st}}{-s}dt\right]=\frac{a}{s}\int_0^{\infty}e^{-st}dt=\frac{a}{s^2}$$

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Table of Laplace Transforms

	$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$		$f(t)=\mathcal{L}^{-1}\{F(s)\}$	$F(s)=\mathcal{L}\{f(t)\}$
1.	1	$\frac{1}{s}$	2.	e^{at}	$\frac{1}{s-a}$
3.	$t^n,\quad n=1,2,3,\ldots$	$\frac{n!}{s^{n+1}}$	4.	$t^p, p>-1$	$\frac{\Gamma(p+1)}{s^{p+1}}$
5.	$\sqrt{t}$	$\frac{\sqrt{\pi}}{2s^{\frac{3}{2}}}$	6.	$t^{n-\frac{1}{2}},\quad n=1,2,3,\ldots$	$\frac{1\cdot3\cdot5\cdots(2n-1)\sqrt{\pi}}{2^n s^{n+\frac{1}{2}}}$
7.	$\sin(at)$	$\frac{a}{s^2+a^2}$	8.	$\cos(at)$	$\frac{s}{s^2+a^2}$
9.	$t\sin(at)$	$\frac{2as}{(s^2+a^2)^2}$	10.	$t\cos(at)$	$\frac{s^2-a^2}{(s^2+a^2)^2}$
11.	$\sin(at)-at\cos(at)$	$\frac{2a^3}{(s^2+a^2)^2}$	12.	$\sin(at)+at\cos(at)$	$\frac{2as^2}{(s^2+a^2)^2}$
13.	$\cos(at)-at\sin(at)$	$\frac{s(s^2-a^2)}{(s^2+a^2)^2}$	14.	$\cos(at)+at\sin(at)$	$\frac{s(s^2+3a^2)}{(s^2+a^2)^2}$
15.	$\sin(at+b)$	$\frac{s\sin(b)+a\cos(b)}{s^2+a^2}$	16.	$\cos(at+b)$	$\frac{s\cos(b)-a\sin(b)}{s^2+a^2}$
17.	$\sinh(at)$	$\frac{a}{s^2-a^2}$	18.	$\cosh(at)$	$\frac{s}{s^2-a^2}$
19.	$e^{at}\sin(bt)$	$\frac{b}{(s-a)^2+b^2}$	20.	$e^{at}\cos(bt)$	$\frac{s-a}{(s-a)^2+b^2}$
21.	$e^{at}\sinh(bt)$	$\frac{b}{(s-a)^2-b^2}$	22.	$e^{at}\cosh(bt)$	$\frac{s-a}{(s-a)^2-b^2}$
23.	$t^n e^{at},\quad n=1,2,3,\ldots$	$\frac{n!}{(s-a)^{n+1}}$	24.	$f(ct)$	$\frac{1}{c}F\left(\frac{s}{c}\right)$
25.	$u_c(t)=u(t-c)$ Heaviside Function	$\frac{e^{-cs}}{s}$	26.	$\delta(t-c)$ Dirac Delta Function	e^{-cs}
27.	$u_c(t)f(t-c)$	$e^{-cs}F(s)$	28.	$u_c(t)g(t)$	$e^{-cs}\mathcal{L}\{g(t+c)\}$
29.	$e^{ct}f(t)$	$F(s-c)$	30.	$t^n f(t),\quad n=1,2,3,\ldots$	$(-1)^n F^{(n)}(s)$
31.	$\frac{1}{t}f(t)$	$\int_s^\infty F(u)\,du$	32.	$\int_0^t f(v)\,dv$	$\frac{F(s)}{s}$
33.	$\int_0^t f(t-\tau)g(\tau)\,d\tau$	$F(s)G(s)$	34.	$f(t+T)=f(t)$	$\frac{\int_0^T e^{-st}f(t)\,dt}{1-e^{-sT}}$
35.	$f'(t)$	$sF(s)-f(0)$	36.	$f''(t)$	$s^2F(s)-sf(0)-f'(0)$
37.	$f^{(n)}(t)$	$s^nF(s)-s^{n-1}f(0)-s^{n-2}f'(0)\cdots-sf^{(n-2)}(0)-f^{(n-1)}(0)$			

Example: Find the Laplace transform of 1+t

Solution:

L (1+t)=1/s + 1/s^2

Example: Find $\mathcal{L}\,[e^{3t}\sin(2t)]$.

Solution: *In this case,* $f(t) = \sin(2t)$

$$F(s) = \mathcal{L}[f(t)] = \frac{2}{s^2+2^2} = \frac{2}{s^2+4}$$

$$\mathcal{L}\left[e^{3t}\sin(2t)\right] = F(s-3) = \frac{2}{(s-3)^2+4}$$

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EXAMPLE

Evaluate **(a)** $\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\}$ **(b)** $\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\}$.

SOLUTION **(a)** we identify $n + 1 = 5$ or $n = 4$ and then multiply and divide by 4!:

$$\mathscr{L}^{-1}\left\{\frac{1}{s^5}\right\} = \frac{1}{4!}\mathscr{L}^{-1}\left\{\frac{4!}{s^5}\right\} = \frac{1}{24}t^4.$$

(b) we identify $k^2 = 7$ and so $k = \sqrt{7}$.

$$\mathscr{L}^{-1}\left\{\frac{1}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\mathscr{L}^{-1}\left\{\frac{\sqrt{7}}{s^2+7}\right\} = \frac{1}{\sqrt{7}}\sin\sqrt{7}t.$$

EXAMPLE

Evaluate $\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\}$.

SOLUTION
$$\mathscr{L}^{-1}\left\{\frac{-2s+6}{s^2+4}\right\} = \mathscr{L}^{-1}\left\{\frac{-2s}{s^2+4} + \frac{6}{s^2+4}\right\}$$

$$= -2\mathscr{L}^{-1}\left\{\frac{s}{s^2+4}\right\} + \frac{6}{2}\mathscr{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

$$= -2\cos 2t + 3\sin 2t.$$

Example 3 Compute the inverse Laplace transform of

$$Y(s) = \frac{2}{3s^4}.$$

Solution $Y(s) = \frac{2}{3s^4} = \frac{1}{9}\cdot\frac{3!}{s^4}$

Thus, by linearity,

$$y(t) = L^{-1}\left[\frac{1}{9}\cdot\frac{3!}{s^4}\right]$$

$$= \frac{1}{9}L^{-1}\left[\frac{3!}{s^4}\right]$$

$$= \frac{1}{9}t^3$$

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