

## Al-Mustagbal University / College of Engineering & Technology **Department: Medical Instrumentation Techniques Engineering**

**Class: Fourth** 

Subject: Control Systems / Code: MU0244002 Lecturer: Asst. Prof. Dr. Hasan Hamad Ali

1<sup>st</sup> term-Lecture No. 3/ Laplace transform-Partial Fractions.



Mathematical background; Laplace transform, complex variable, matrices.

### **Partial Fractions**

There are different types of proper fractions, depending on the specific characteristics of the numerator and denominator as follows:

#### 1. Non-repeated Linear Factor

$$\frac{N(s)}{(as+b)(cs+d)} = \frac{A}{(as+b)} + \frac{B}{(cs+d)}$$

**Ex.1:** If 
$$Y(s) = \frac{(s+2)(s+4)}{s(s+1)(s+3)}$$
, find  $y(t)$ 

**Solution.** We may write Y(s) in terms of its partial-fraction expansion:

$$Y(s) = \frac{C_1}{s} + \frac{C_2}{s+1} + \frac{C_3}{s+3}$$

Using the cover-up method, we get

$$C_1 = \frac{(s+2)(s+4)}{(s+1)(s+3)}\Big|_{s=0} = \frac{8}{3}$$

In a similar fashion.

$$C_2 = \frac{(s+2)(s+4)}{s(s+3)} \Big|_{s=-\frac{3}{2}}$$

and

$$C_3 = \frac{(s+2)(s+4)}{s(s+1)}\Big|_{s=-3} = -\frac{1}{6}$$

$$y(t) = \frac{8}{3} - \frac{3}{2}e^{-t} - \frac{1}{6}e^{-3t}$$

## 2. Repeated Linear Factor

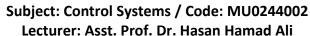
$$\frac{N(s)}{(as+b)^2} = \frac{A}{(as+b)} + \frac{B}{(as+b)^2}$$

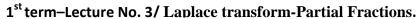
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**Ex. 2:** If 
$$Y(s) = \frac{(s+3)}{(s+1)(s+2)^2}$$
, find  $y(t)$ 

Solution. We write the partial fraction as

$$F(s) = \frac{C_1}{s+1} + \frac{C_2}{s+2} + \frac{C_3}{(s+2)^2}.$$

Then

$$C_{1} = (s+1)F(s)|_{s=-1} = \frac{s+3}{(s+2)^{2}}\Big|_{s=-1} = 2,$$

$$C_{2} = \frac{d}{ds}(s+2)^{2}F(s)|_{s=-2} = -2,$$

$$C_{3} = (s+2)^{2}F(s)|_{s=-2} = \frac{s+3}{s+1}\Big|_{s=-2} = -1.$$

$$y(t) = 2e^{-t} - 2e^{-2t} - te^{-2t}$$

Ex.3: If 
$$Y(s) = \frac{(s^2+3)}{s(s+2)^2}$$
, find  $y(t)$ 

Sol. 
$$\frac{(s^2+3)}{s(s+2)^2} = \frac{A}{s} + \frac{B}{s+2} + \frac{C}{(s+2)^2} = \frac{A(s+2)^2 + Bs(s+2) + Cs}{s(s+2)^2}$$

$$(s^2 + 3) = A(s + 2)^2 + Bs(s + 2) + Cs$$

$$A=3/4$$
 ,  $B=1/4$  ,  $C=-7/2$ 

Then, 
$$\frac{(s^2+3)}{s(s+2)^2} = \frac{3/4}{s} + \frac{1/4}{s+2} - \frac{7/2}{(s+2)^2}$$

$$y(t) = \frac{3}{4} + \frac{1}{4}e^{-2t} - \frac{7}{2}te^{-2t}$$

#### 3. Quadratic Factor

$$\frac{(3s^2 - 4)}{s(s^2 + 1)} = \frac{A}{s} + \frac{Bs + C}{s^2 + 1} = \frac{A(s^2 + 1) + (Bs + C)s}{s(s^2 + 1)}$$

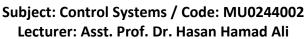
$$3s^2 - 4 = A(s^2 + 1) + (Bs + C)s$$

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$$A=4$$
 ,  $B=-1$  ,  $C=0$ 

Then, 
$$\frac{(3s^2-4)}{s(s^2+1)} = \frac{4}{s} - \frac{s}{s^2+1}$$

$$y(t) = 4 - cost$$

Compute the inverse Laplace transform of

$$Y(s) = \frac{2s+5}{s^2+8s+12} = \frac{2s+5}{(s+2)(s+6)}$$

Clearly, the two poles (s = -2, s = -6) are distinct. Therefore, the partial-fraction expansion is

$$Y(s) = \frac{2s+5}{(s+2)(s+6)} = \frac{a_1}{s+2} + \frac{a_2}{s+6}$$

Using Eq. (8.16), the first residue is

$$a_1 = (s+2)Y(s)|_{s=-2} = \frac{2s+5}{s+6}\Big|_{s=-2} = \frac{1}{4} = 0.25$$

$$a_2 = (s+6)Y(s)|_{s=-6} = \frac{2s+5}{s+2}\Big|_{s=-6} = \frac{-7}{-4} = 1.75$$

Using the residues, the partial-fraction expansion is

$$Y(s) = \frac{0.25}{s+2} + \frac{1.75}{s+6}$$

$$y(t) = 0.25e^{-2t} + 1.75e^{-6t}$$