Al-Mustaqbal University

College of engineering and technology

chemical and petroleum industrial Department

class –three Term-1

Heat transfer – Code No. UOMU0102051

week-3 General conduction equation with heat generation

Prof. Dr. Majid H. Hameed

General Conduction equation

The general equations of heat conduction in three dimensions in different type of coordinates are:

1. Cartesian coordinates

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$ Fourier-Biot equation

Where: $\alpha=\frac{k}{\rho C}$ the thermal diffusivity

a) For steady state

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{g}}{k}=0$$ Poisson equation

b) Transient, and with no heat generation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$ Diffusion equation

c) Steady, and with no heat generation

$$\frac{\partial^2 T}{\partial x^2}+\frac{\partial^2 T}{\partial y^2}+\frac{\partial^2 T}{\partial z^2}=0$$

General Conduction equation

Note: All these equation can be written in two dimensions and one dimension by omitting one or two terms of differentiations of x, y, z.

2. Cylindrical Coordinates

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{1}{r^2}\frac{\partial^2 T}{\partial \emptyset^2}+\frac{\partial^2 T}{\partial z^2}+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

3. Spherical Coordinates

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial T}{\partial r}\right)+\frac{1}{r^2\,sin^2\,\emptyset}\frac{\partial^2 T}{\partial \emptyset^2}+\frac{1}{r^2\,sin^2\,\theta}\frac{\partial^2 T}{\partial \theta^2}+\frac{\dot{g}}{k}=\frac{1}{\alpha}\frac{\partial T}{\partial \tau}$$

General Conduction equation

The temperature distribution equation:

The temperature distribution equation in three dimensions and two dimensions are difficult to solve by hand so it is leave for post graduations. The equation of one-dimensional temperature distribution can be solved in this level of study.

1. **Study-state with no heat generation**

As it is taken in the first week

Differential equation of heat conduction in a slab in steady state with no heat generation is:

$$\frac{d}{dx}\left(k\frac{dT}{dx}\right) = 0$$

For k=constant, the equation becomes

General Conduction equation

- For k=constant, the equation becomes

$$\frac{d^2T}{dx^2} = 0$$

Boundary conditions

B.C.1 At $x=x_1$ $T=T_1$, B.C.2 at $x=x_2$ $T=T_2$

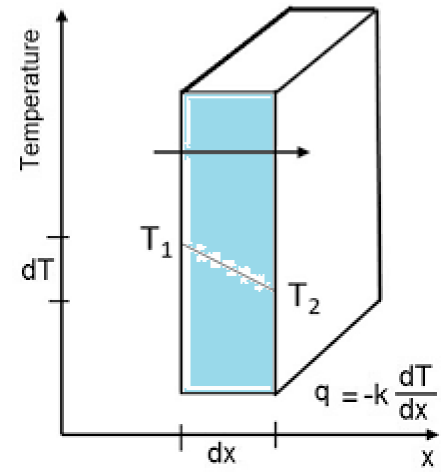
By first integration $\frac{dT}{dx} = C_1$

By second integration $T = C_1x + C_2$ (B.C.E)

C_1, C_2 are the integration constants

By applying the B.C.1 $T_1 = C_1x_1 + C_2$ (1)

By applying the B.C.2 $T_2 = C_1x_2 + C_2$ (2)



General Conduction equation

By abstract eq.(2) from eq.(1) $(T_1 - T_2) = C_1(x_1 - x_2)$,

$C_1 = \frac{(T_1 - T_2)}{(x_1 - x_2)}$

By substituting in eq.(1) $T_1 = C_1 x_1 + C_2 = \frac{(T_1 - T_2)}{(x_1 - x_2)} x_1 + C_2$

Then, $C_2 = T_1 - \frac{(T_1 - T_2)}{(x_1 - x_2)} x_1$

By substituting C_1 and C_2 in the (T.D.E)

$$T = C_1 x + C_2 = \frac{(T_1 - T_2)}{(x_1 - x_2)} x + T_1 - \frac{(T_1 - T_2)}{(x_1 - x_2)} x_1$$

$$T - T_1 = \frac{(T_1 - T_2)}{(x_1 - x_2)} (x - x_1) = (T_2 - T_1) \frac{(x - x_1)}{(x_2 - x_1)}$$

$$\boldsymbol{\frac{(T - T_1)}{(T_2 - T_1)} = \frac{(x - x_1)}{(x_2 - x_1)} = \frac{(x - x_1)}{\Delta x}}$$

General Conduction equation

Ex.1. A slab is of thickness 5cm. The temperature at one side is 80^oC and the temperature at the other side is 20^oC. Find the temperature distribution equation and find the temperature at the midpoint of the slab.

Solution: a slab Δx=5cm, $x_1 = 0$ $T_1=80^oC$, and x_2=5cm=0.05cm, $T_2=20^oC$

Requirements: T.D.E, Temperature of midpoint of the slab

Analysis: The boundary conditions are:

General Conduction equation

B.C.1 $x_1 = 0$ T_1=80°C, and B.C.2 x_2=5cm=0.05cm, T_2=20°C

The differential equation is: $\frac{d^2T}{dx^2} = 0$

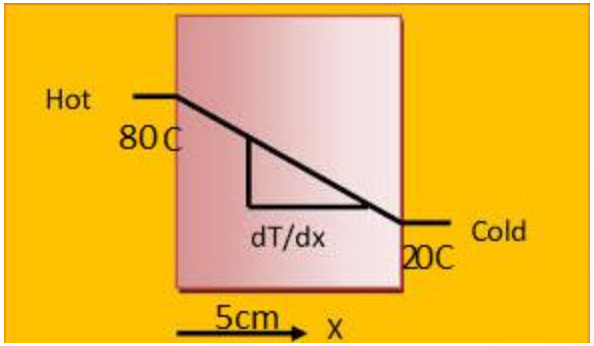
After double integrations of this relation $T = C_1 x + C_2$

And applying the B.Cs we find the T.D.E

$$\frac{(T-T_1)}{(T_2-T_1)} = \frac{(x-x_1)}{\Delta x} \quad \text{or} \quad \frac{(T-80)}{(20-80)} = \frac{(x-0)}{5}$$

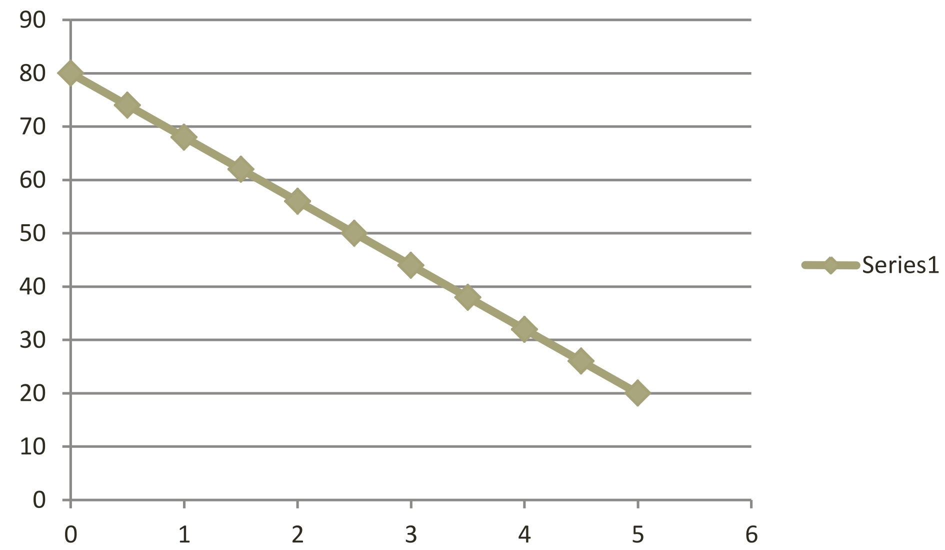
$$T = 80 - \frac{60x}{5}$$

$$T = 80 - 12x$$



General Conduction equation

X	0	0.5	1.0	1.5	2	2.5	3.0	3.5	4.0	4.5	5.0	Cm
T	80	74	68	62	56	50	44	38	32	26	20	°C



General Conduction equation

2. Plane wall with heat source.

The heat source or heat generation is a uniform heat is generating in body its unit W/m^3

The differential equation of temperature with heat generation and study state is to be as below

$\frac{d}{dx}\left(k\frac{dT}{dx}\right) + \dot{g} = 0$ $\dot{g}$: The heat generation or heat source. W/m^3

If the thermal conductivity is constant (k=constant) the relation becomes

$$\frac{d^2T}{dx^2} + \frac{\dot{g}}{k} = 0, \text{ Or } \qquad \boldsymbol{\frac{d^2T}{dx^2} = -\frac{\dot{g}}{k}}$$

General Conduction equation

To solve relation we need description of the case of the body and its boundary conditions.

We will take a slab of thickness L, its temperature of its two ends are T_1, and T_2. There is heat generation of $\dot{g}$ W/m^3. The slab has constant thermal conductivity k. The slab is shown in the figure.

$$\frac{d^2T}{dx^2} = -\frac{\dot{g}}{k}$$

The boundary conditions are

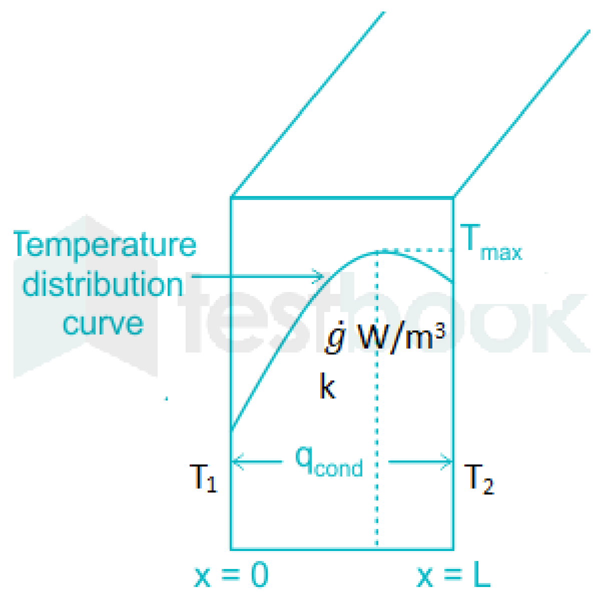
B.C.1 at x=0 $T=T_1$

B.C.2 at x=L $T=T_2$

By integration the equation

$$\frac{d^2T}{dx^2} = -\frac{\dot{g}}{k}$$

$$\frac{dT}{dx} = -\frac{\dot{g}}{k}x + C_1$$



General Conduction equation

And by second integration

$$T = -\frac{\dot{g}x^2}{2k} + C_1 x + C_2 \qquad \text{(T.D.E)}$$

By applying the boundary conditions we get:

$$T_1 = 0 + 0 + C_2 \rightarrow C_2 = T_1$$

$$T = -\frac{\dot{g}x^2}{2k} + C_1 x + T_1$$

$$T_2 = -\frac{\dot{g}L^2}{2k} + C_1 L + T_1 \rightarrow C_1 = \frac{T_2 - T_1}{L} + \frac{\dot{g}L}{2k}$$

Then $T = -\frac{\dot{g}x^2}{2k} + \left(\frac{T_2 - T_1}{L} + \frac{\dot{g}L}{2k}\right) x + T_1$

$$T = \frac{\dot{g}L}{2k} x - \frac{\dot{g}x^2}{2k} + \frac{x}{L}(T_2 - T_1) + T_1$$

$$\boldsymbol{T = \frac{\dot{g}L}{2k}\left(x - \frac{x^2}{L}\right) + \frac{x}{L}(T_2 - T_1) + T_1} \qquad \text{(T.D.E)}$$

General Conduction equation

The heat flux at any surface can be calculated as below

$$\dot{q} = -k\frac{dT}{dx}$$

The heat flux at x=0

$$\dot{q}_{x=0} = -k\left.\frac{dT}{dx}\right)_{x=0} = -k\frac{d}{dx}\left(\frac{\dot{\boldsymbol{g}}\boldsymbol{L}}{\boldsymbol{2k}}\left(\boldsymbol{x}-\frac{\boldsymbol{x}^2}{\boldsymbol{L}}\right)+\frac{\boldsymbol{x}}{\boldsymbol{L}}(\boldsymbol{T_2}-\boldsymbol{T_1})+\boldsymbol{T_1}\right)_{x=0}$$

$$\dot{q}_{x=0} = -k\left(\frac{\dot{g}L}{2k}\left(1-\frac{2x}{L}\right)+\frac{1}{L}(T_2-T_1)\right)_{x=0}$$

$$\dot{q}_{x=0} = -\left(\frac{\dot{g}L}{2}\left(1-\frac{2\times 0}{L}\right)+\frac{k}{L}(T_2-T_1)\right) = -\left(\frac{\dot{g}L}{2}+\frac{k}{L}(T_2-T_1)\right)$$

- The heat flux at x=L

$$\dot{q}_{x=0} = -k\left.\frac{dT}{dx}\right)_{x=L} = -k\frac{d}{dx}\left(\frac{\dot{\boldsymbol{g}}\boldsymbol{L}}{\boldsymbol{2k}}\left(\boldsymbol{x}-\frac{\boldsymbol{x}^2}{\boldsymbol{L}}\right)+\frac{\boldsymbol{x}}{\boldsymbol{L}}(\boldsymbol{T_2}-\boldsymbol{T_1})+\boldsymbol{T_1}\right)_{x=L}$$

$$\dot{q}_{x=L} = -k\left(\frac{\dot{g}L}{2k}\left(1-\frac{2x}{L}\right)+\frac{1}{L}(T_2-T_1)\right)_{x=L}$$

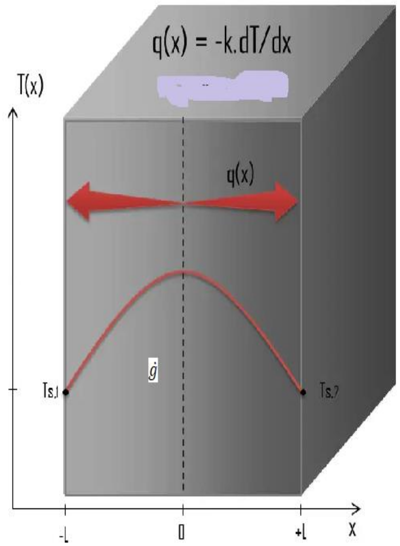
$$\dot{q}_{x=L} = -\left(\frac{\dot{g}L}{2}\left(1-\frac{2\times L}{L}\right)+\frac{k}{L}(T_2-T_1)\right) = \left(\frac{\dot{g}L}{2}-\frac{k}{L}(T_2-T_1)\right)$$

General Conduction equation

Ex.2. Take a wall of constant thermal conductivity and uniform heat generation of $\dot{g}$. The wall surfaces temperature is the same and equal T_w. Find the equation of the temperature distribution and the heat flux at its surfaces.

Solution: assume the thickness of the wall is 2L,

- The boundary conditions are:
- B.C.1 $T=T_w$ at $x=\mp L$, and B.C.2 $\frac{dT}{dx}=0$ at x=0
- $\frac{d^2T}{dx^2}=-\frac{\dot{g}}{k}$
- By first integration we get
- $\frac{dT}{dx}=-\frac{\dot{g}x}{k}+C_1$
- From B.C.2 we find $C_1=0$
- The equation becomes
- $\frac{dT}{dx}=-\frac{\dot{g}x}{k}$



General Conduction equation

By second integration

$$T = -\frac{\dot{g}x^2}{2k} + C_2 \quad \text{(T.D.E)}$$

By applying the first boundary condition

$$C_2 = T_w + \frac{\dot{g}L^2}{2k}$$

Then $T = -\frac{\dot{g}x^2}{2k} + T_w + \frac{\dot{g}L^2}{2k}$

$$\boldsymbol{T - T_w = \frac{\dot{g}L^2}{2k}\left(1 - \left(\frac{x}{L}\right)^2\right)}$$

$T_o - T_w = \frac{\dot{g}L^2}{2k}$ at the midline of the wall

$$\frac{T - T_w}{T_o - T_w} = \left(1 - \left(\frac{x}{L}\right)^2\right)$$

To find the heat flux at x=-L

General Conduction equation

$$\dot{\boldsymbol{q}}_{\boldsymbol{x=-L}} = -k\frac{dT}{dx}\bigg)_{x=-L} = -k\frac{d}{dx}\left(T_w + \frac{\dot{g}L^2}{2k}\left(1-\left(\frac{x}{L}\right)^2\right)\right)_{x=-L}$$

$$= -k\left(-\frac{\dot{g}}{2k}(2x)\right)_{x=-L} = -\dot{g}L$$

And $\dot{\boldsymbol{q}}_{\boldsymbol{x=L}} = \dot{\boldsymbol{g}}\boldsymbol{L}$

General Conduction equation

Cylinder without heat source

take a hollow cylinder

Boundary conditions: at $r=r_1$ $T=T_1$, at $r=r_2$ $T=T_2$

$$\frac{d}{dr}\left(kr\frac{dT}{dr}\right)=0$$

If k=constant

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=0$$

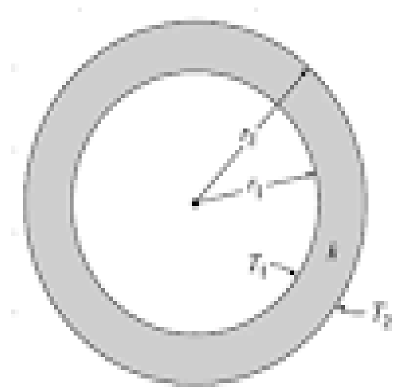
By first integration

$$r\frac{dT}{dr}=C_1 \rightarrow \frac{dT}{dr}=\frac{C_1}{r}$$

By second integration

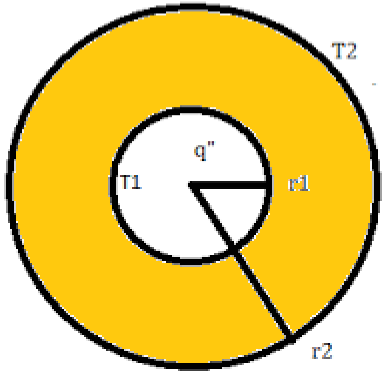
$$T=C_1 lnr+C_2$$

By applying the B.Cs



General Conduction equation

- $T = C_1 lnr + C_2$
- By applying the B.Cs
- $T_1 = C_1 lnr_1 + C_2$
- $T_2 = C_1 lnr_2 + C_2$
- The solution is
- $\frac{T - T_1}{T_2 - T_1} = \frac{ln\left(\frac{r}{r_1}\right)}{ln\left(\frac{r_2}{r_1}\right)}$



General Conduction equation

Solid Cylinder with heat source

We can take solid cylinder of constant thermal conductivity k and uniform heat generation $\dot{g}$ as shown in the figure.

The B.Cs are B.C.1 r=0, $\frac{dT}{dr} = 0$, B.C.2 $r=r_o$ $T=T_w$

Differential equation with heat generation

General Conduction equation

$$\frac{d}{dr}\left(rk\frac{dT}{dr}\right)+\dot{g}r=0$$

Where k=constant

$$\frac{d}{dr}\left(r\frac{dT}{dr}\right)=-\frac{\dot{g}r}{k}$$

By the first integration

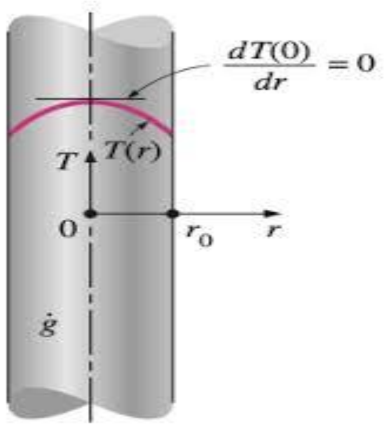
$$\left(r\frac{dT}{dr}\right)=-\frac{\dot{g}r^2}{2k}+C_1$$

By applying the first B.C, $0=-0+C_1 \rightarrow C_1=0$

$$\left(r\frac{dT}{dr}\right)=-\frac{\dot{g}r^2}{2k}\rightarrow\frac{dT}{dr}=-\frac{\dot{g}r}{2k}$$

By second integration we get

$$T=-\frac{\dot{g}r^2}{4k}+C_2 \qquad \text{(T.D.E)}$$



General Conduction equation

By applying second B.C.2

$$T_o = -\frac{\dot{g}{r_o}^2}{4k} + C_2 \quad \rightarrow C_2 = T_w + \frac{\dot{g}{r_o}^2}{4k}$$

$$T = -\frac{\dot{g}r^2}{4k} + T_w + \frac{\dot{g}{r_o}^2}{4k}$$

$$T - T_w = \frac{\dot{g}{r_o}^2}{4k}\left(1 - \frac{r^2}{{r_o}^2}\right)$$

$$T_o - T_w = \frac{\dot{g}{r_o}^2}{4k}$$ At the center line of the cylinder

$$\frac{T-T_w}{T_o-T_w} = \left(1 - \frac{r^2}{{r_o}^2}\right)$$

$$\dot{q}_{r=r_o} = -k\left.\frac{dT}{dr}\right)_{r=r_o} = -k\left(-\frac{\dot{g}r}{2k}\right)_{r=r_o} = \frac{\dot{g}r_o}{2}$$

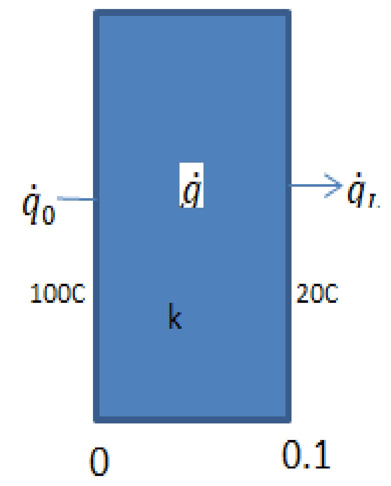
General Conduction equation

Ex.3. A plane wall of 10cm thick one surface of the wall is at temperature of 100^oC and the other surface is at 20^oC. Thermal conductivity of the wall material is 30W/m.^oC and the heat generation in the wall is 6x10^{5}W/m^3. Find the temperature distribution equation and the location and magnitude of maximum temperature in the wall and the heat flux from its two surfaces.

Solution: A plane wall L=10cm=0.1m, at x=0 T=100^oC, and at x=0.1m T=20^oC, k=30W/m.^oC, $\dot{g} = 6 \times 10^5 W/m^3$

General Conduction equation

Requirements: temperature distribution equation
The magnitude and location
of maximum temperature
Heat flux at each surfaces of
the wall.



Analysis: to solve this problem,
we can put the boundary condition
as below
$x = 0, T = T_1 = 100$°C, and x=L, T=T_2=20°C
The differential of temperature distribution is

$$\frac{d^2T}{dx^2} = -\frac{\dot{g}}{k}$$

General Conduction equation

The first integrating gives $\frac{dT}{dx} = -\frac{\dot{g}x}{k} + C_1$

The second integration gives $T = -\frac{\dot{g}x^2}{2k} + C_1 x + C_2$

By applying B.C.1 $T_1 = -\frac{\dot{g}(0)^2}{2k} + C_1(0) + C_2 \rightarrow C_2 = T_1 = 100$°C

From B.C.2 $T_2 = -\frac{\dot{g}(L)^2}{2k} + C_1 L + T_1$

$$\left(\frac{T_2 - T_1}{L}\right) + \frac{\dot{g}L}{2k} = C_1 \rightarrow C_1 = \frac{20-100}{0.1} + \frac{6\times10^5\times0.1}{2\times30} = -800 + 1000 = 200$$

$$T = -\frac{\dot{g}x^2}{2k} + \left[\left(\frac{T_2 - T_1}{L}\right) + \frac{\dot{g}L}{2k}\right] x + T_1$$

$$T = -\frac{6\times10^5\times x^2}{2\times30} + [400]x + 100 = -10000x^2 + 200x + 100 \quad \text{T.D.E.}$$

General Conduction equation

- To find the location and magnitude of the maximum temperature
- $\frac{dT}{dx} = -20000x + 200 = 0 \rightarrow x = 0.01m$
- $T_{max} = -10000(0.01)^2 + 200 \times 0.01 + 100 = 100$

X	0	0.01 Tmax	0.02	0.04	0.06	0.08	0.1
T	100	101	100	92	76	52	20

General Conduction equation

The heat flux at the two surfaces

$$\dot{q}_x = -k\frac{dT}{dx}$$

$$\dot{q}_0 = -k\left.\frac{dT}{dx}\right)_{x=0} = -k\frac{d}{dx}\left[-\frac{\dot{g}x^2}{2k} + \left[\left(\frac{T_2-T_1}{L}\right) + \frac{\dot{g}L}{2k}\right]x + T_1\right]_{x=0}$$

$$\dot{q}_0 = -k\left[-\frac{\dot{g}x}{k} + \left(\frac{T_2-T_1}{L}\right) + \frac{\dot{g}L}{2k}\right]_{x=0} = -k\left[\left(\frac{T_2-T_1}{L}\right) + \frac{\dot{g}L}{2k}\right]$$

$$= -30\left[\left(\frac{20-100}{0.1}\right) + \frac{6\times10^5\times0.1}{2\times30}\right] = -30(-800 + 1000)$$

$$= -6000\text{W/m}^2$$

General Conduction equation

$$\dot{q}_L = -k\frac{dT}{dx}\Big)_{x=L} = -k\frac{d}{dx}\left[-\frac{\dot{g}x^2}{2k} + \left[\left(\frac{T_2-T_1}{L}\right) + \frac{\dot{g}L}{2k}\right]x + T_1\right]_{x=L}$$

$$\dot{q}_L = -k\left[-\frac{\dot{g}x}{k} + \left(\frac{T_2-T_1}{L}\right) + \frac{\dot{g}L}{2k}\right]_{x=L} = -k\left[\left(\frac{T_2-T_1}{L}\right) - \frac{\dot{g}L}{2k}\right]$$

$$= -30\left[\left(\frac{20-100}{0.1}\right) - \frac{6\times10^5\times0.1}{2\times30}\right] = 54000\ \mathrm{W/m^2}$$

$\dot{g}\times L = \dot{q}_o + \dot{q}_L$ in magnitude

General Conduction equation

Ex.4. **Long solid cylinder of diameter 100cm is of thermal conductivity 20W/m.°C, and heat generation $3x10^4 W/m^3$. If the temperature of the out surface is 30°C. Find the temperature distribution equation and heat flux.**

Solution: long solid cylinder D=100cm=1m, r_o=0.5m, k=20W/m.°C, $\dot{g} = 3 \times 10^4 W/m^3$. T=$T_o$=30°C at r=$r_o$=0.5

Requirements: Temperature distribution equation and heat flux at the surface.

Analysis: The boundary condition

- At r=0 $\frac{dT}{dr} = 0$ and at r=r_o T=T_o

General Conduction equation

- The differential equation $\frac{d}{dr}\left(r\frac{dT}{dr}\right) = -\frac{\dot{g}r}{k}$
- First integrating gives $r\frac{dT}{dr} = -\frac{\dot{g}r^2}{2k} + C_1$ from B.C.1 C_1=0.0
- Then $r\frac{dT}{dr} = -\frac{\dot{g}r^2}{2k} \rightarrow \frac{dT}{dr} = -\frac{\dot{g}r}{2k}$
- Second integrating gives $T = -\frac{\dot{g}r^2}{4k} + C_2,$ from B.C.2

  $T_o = -\frac{\dot{g}{r_o}^2}{4k} + C_2$
- $C_2 = T_o + \frac{\dot{g}{r_o}^2}{4k}$

General Conduction equation

Then $T = T_o + \frac{\dot{g}{r_o}^2}{4k} - \frac{\dot{g}r^2}{4k} = T_o + \frac{\dot{g}}{4k}({r_o}^2 - r^2)$

$$T = T_o + \frac{\dot{g}}{4k}({r_o}^2 - r^2) = 30 + \frac{10^4}{4\times20}[(0.5)^2 - r^2]$$

T.D.E.

$T = 30 + 125[0.25 - r^2]$

r	0	0.05	0.1	0.15	0.2	0.25	0.3	0.35	0.4	0.45	0.5
T	61.25	60.9375	60	58.4375	56.25	53.4375	50	45.9375	41.25	35.9375	30

General Conduction equation

- The heat flux at outer surface is
- $\dot{q}_{r=r_o} = -k\left.\frac{dT}{dr}\right)_{r=r_o} = -k\frac{d}{dr}\left[T_o + \frac{\dot{g}}{4k}({r_o}^2 - r^2)\right]\Big)_{r=r_o}$
  $= -k\left[\frac{\dot{g}}{4k}(-2r)\right]\Big)_{r=r_o}$
- $\dot{q}_{r=r_o} = \frac{\dot{g}r_o}{2} = \frac{10^4 \times 0.5}{2} = 2500W/m^2$