Functions: Domain and Range

The temperature at which water boils depends on the elevation above sea level (the boiling point drops as you ascend). The interest paid on a cash investment depends on the length of time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels from an initial location along a straight line path depends on its speed.

In each case, the value of one variable quantity, which we might call y, depends on the value of another variable quantity, which we might call *x*. Since the value of *y* is completely determined by the value of *x*, we say that *y* is a function of *x*. Often the value of y is given by a rule or formula that says how to calculate it from the variable *x*. For instance, the equation $A = \pi r^2$ is a rule that calculates the area *A* of a circle from its radius r.

In calculus we may want to refer to an unspecified function without having any particular formula in mind. A symbolic way to say "*y* is a function of *x*" is by writing: ***y = f(x)*** *("y equals f of x").*

In this notation, the symbol ***f*** represents the function. The letter ***x***, called the independent variable, represents the input value of f, and y, the dependent variable, represents the corresponding output value of ***f*** at *x*

DEFINITION Function

A **function** from a set D to a set Y is a rule that assigns a *unique* (single) element $f(x) \in Y$ to each element $x \in D$.

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The set *D* of all possible input values is called the domain of the function. The set of all values of *f(x)* as *x* varies throughout *D* is called the range of the function. The range may not include every element in the set *Y*

EXAMPLE 1 Identifying Domain and Range

Verify the domains and ranges of these functions.

Function	Domain (x)	Range (y)
$y = x^2$	$(-\infty, \infty)$	$[0, \infty)$
$y = 1/x$	$(-\infty, 0) \cup (0, \infty)$	$(-\infty, 0) \cup (0, \infty)$
$y = \sqrt{x}$	$[0, \infty)$	$[0, \infty)$
$y = \sqrt{4 - x}$	$(-\infty, 4]$	$[0, \infty)$
$y = \sqrt{1 - x^2}$	$[-1, 1]$	$[0, 1]$

H.W

In Exercises 1–6, find the domain and range of each function.

1. $f(x) = 1 + x^2$

2. $f(x) = 1 - \sqrt{x}$

3. $F(t) = \dfrac{1}{\sqrt{t}}$

4. $F(t) = \dfrac{1}{1 + \sqrt{t}}$

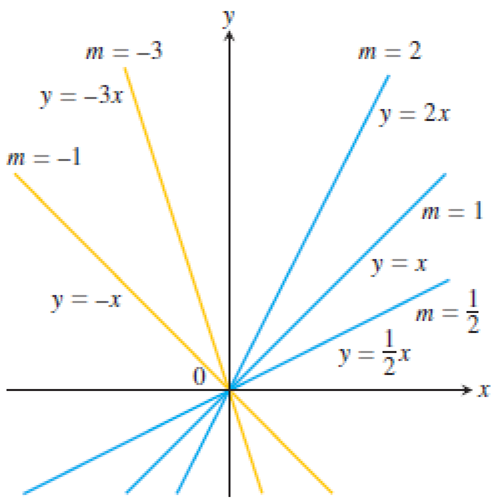
5. $g(z) = \sqrt{4 - z^2}$

6. $g(z) = \dfrac{1}{\sqrt{4 - z^2}}$

Identifying Functions; Mathematical Models

There are a number of important types of functions frequently encountered in calculus. We identify and briefly summarize them here.

Linear Functions A function of the form $f(x) = mx + b$, for constants m and b, is called a **linear function**. Figure 2 shows an array of lines $f(x) = mx$ where $b = 0$, so these lines pass through the origin. Constant functions result when the slope $m = 0$ (Figure 3).



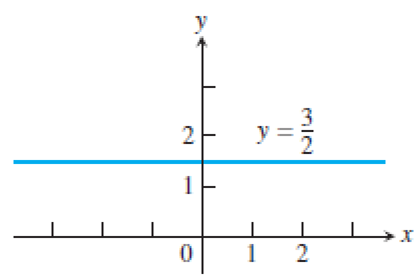


FIGURE 2: The collection of lines $y = mx$ has slope m and all lines pass through the origin.

FIGURE 3: The collection of lines $y = mx$ has slope m and all lines pass through the origin.

__Power Functions__ A function $f(x) = x^a$, where a is a constant, is called a **power function**. There are several important cases to consider

a) $a = n \quad positive\ integer.$

The graphs of $f(x) = x^n$, for $n = 1, 2, 3, 4, 5$, are displayed in Figure 4. These functions are defined for all real values of x. Notice that as the power n gets larger,

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the curves tend to flatten toward the x-axis on the interval s -1, 1d, and also rise more steeply for $|x| > 1$. Each curve passes through the point (1, 1) and through the origin.

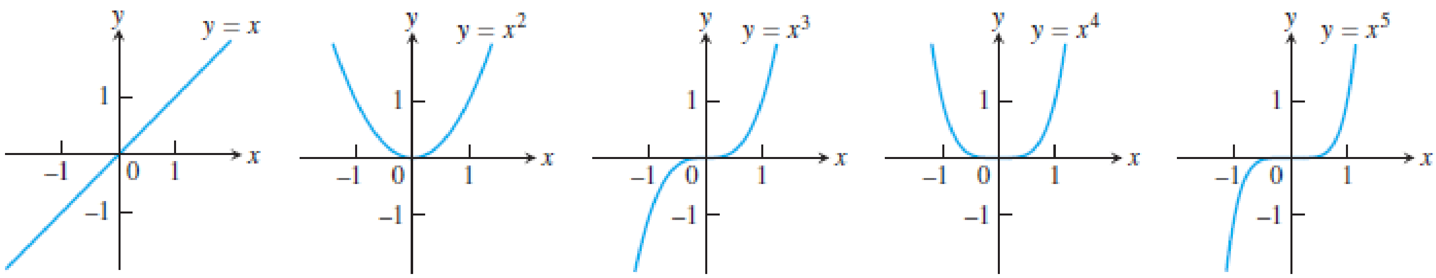

FIGURE 4: Graphs of $f(x) = x^n$, $n = 1, 2, 3, 4, 5$

b) $\boldsymbol{a = -1 \quad or \quad a = -2}$

The graphs of the functions $f(x) = x^{-1} = 1/x$ and $g(x) = x^{-2} = 1/x^2$ are shown in Figure 5. Both functions are defined for all $x \neq 0$(you can never divide by zero).

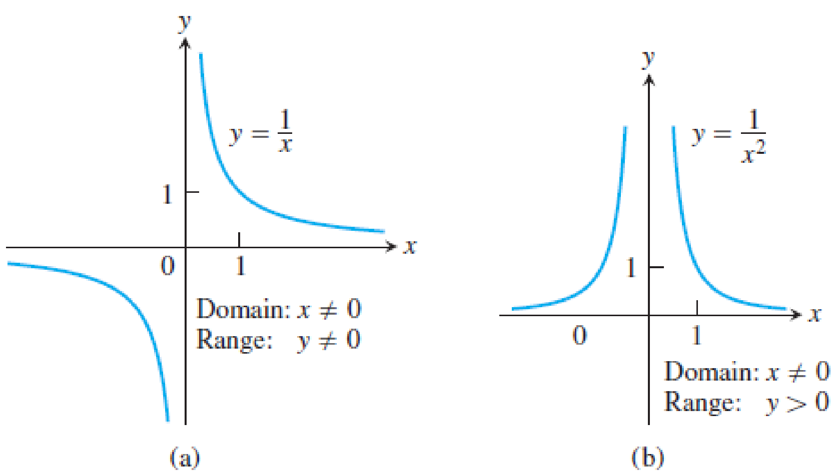

FIGURE 5: Graphs of the power functions $f(x) = x^a$ for part (a) a = -1 and for part (b) a = -2.

c) $\boldsymbol{a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2} \; and \; \frac{2}{3}}$

The functions $f(x) = x^{1/2} = \sqrt{x}$ and $g(x) = x^{1/3} = \sqrt[3]{x}$ x are the **square root** and **cube root** functions, respectively. The domain of the square root function is $[0, \infty)$ but the cube root function is defined for all real x. Their graphs are displayed in Figure 6 along with the graphs of $y = x^{3/2}$ and $y = x^{2/3}$.

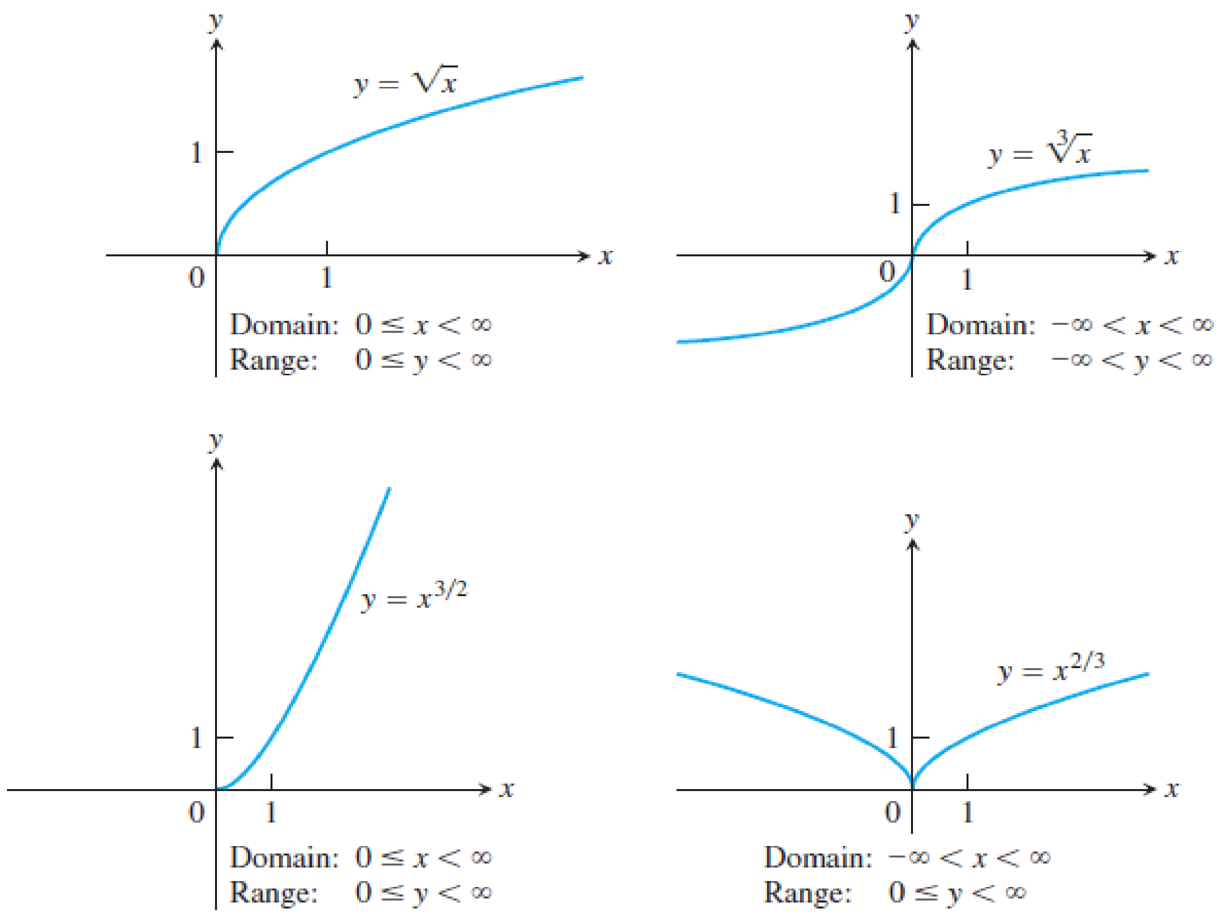

FIGURE 6: Graphs of the power functions $f(x) = x^a$ for $a = \frac{1}{2}, \frac{1}{3}, \frac{3}{2}$ and $\frac{2}{3}$

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Polynomials A function p is a polynomial if

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

where n is a nonnegative integer and the numbers a_0 , a_1 , a_2 , ….., a_n are real constants (called the coefficients of the polynomial). All polynomials have domain $(-\infty, \infty)$. If the leading coefficient $a_n \neq 0$ and $n>0$, then n is called the degree of the polynomial. Linear functions with $m \neq 0$ are polynomials of degree 1. Polynomials of degree 2, usually written as $p(x)= ax^2 + bx + c$, are called quadratic functions. Likewise, cubic functions are polynomials $p(x)= ax^3 + bx^2 + cx + d$ of degree 3. Figure 7 shows the graphs of three polynomials.

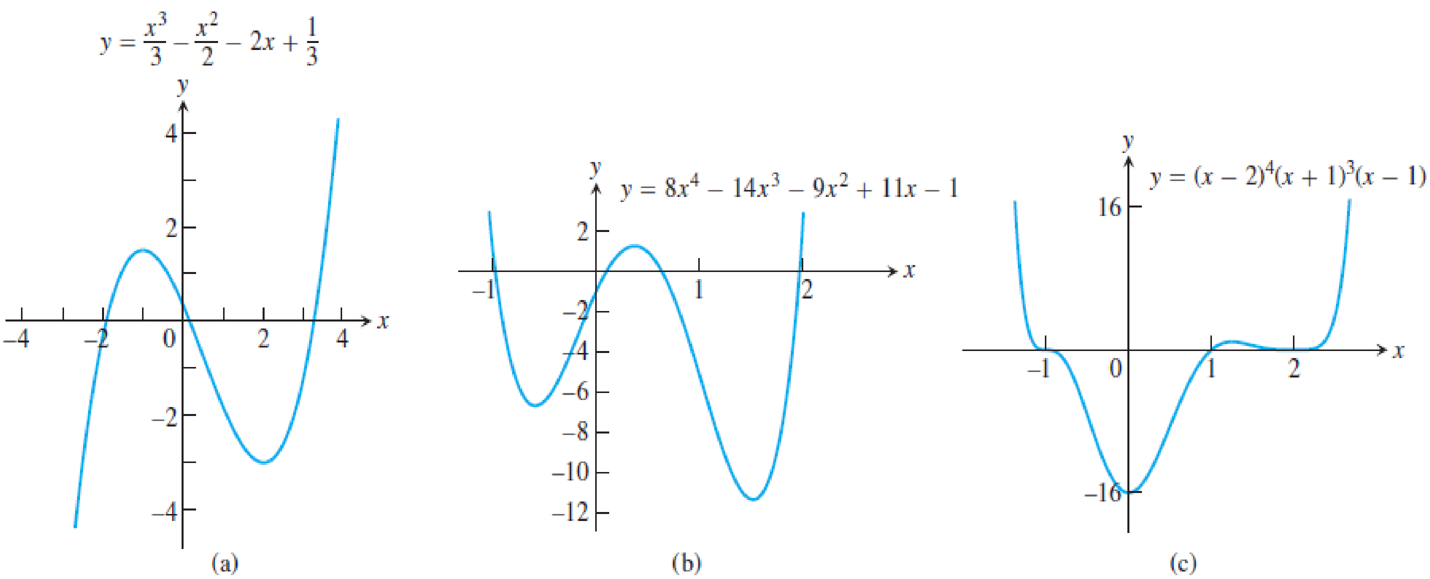

FIGURE 7: Graphs of three polynomial functions.

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Rational Functions: A **rational function** is a quotient or ratio of two polynomials:

$$f(x) = \frac{p(x)}{q(x)}$$

where p and q are polynomials. The domain of a rational function is the set of all real x for which $q(x) \neq 0$. For example, the function

$$f(x) = \frac{2x^2 - 3}{7x + 4}$$

is a rational function . Its graph is shown in Figure 8a with the graphs of two other rational functions in Figures 8b and 8c.

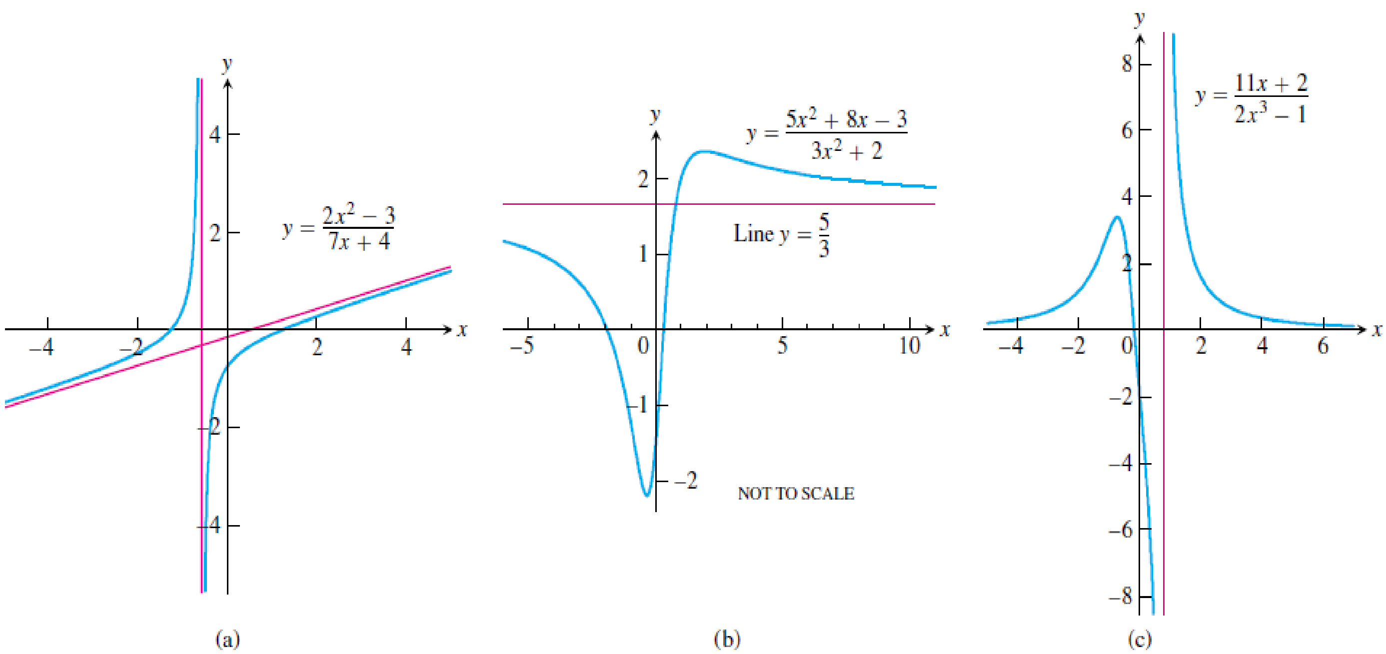

FIGURE 8: Graphs of three rational functions

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Algebraic Functions An **algebraic function** is a function constructed from polynomials using algebraic operations (addition, subtraction, multiplication, division, and taking roots). Rational functions are special cases of algebraic functions. Figure 9 displays the graphs of three algebraic functions.

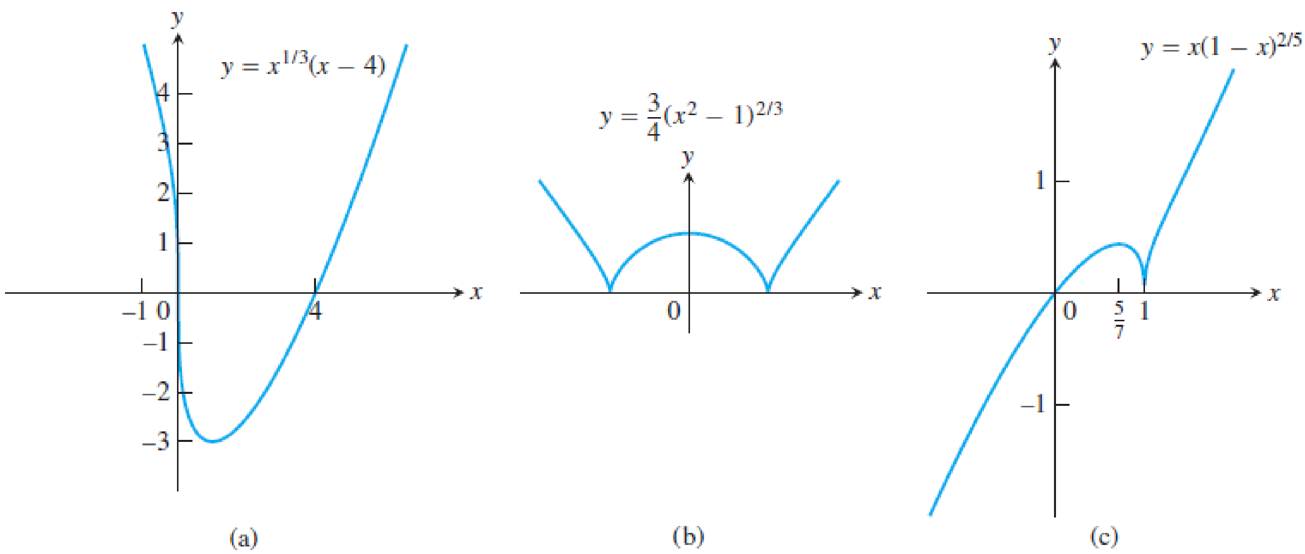

FIGURE 9: Graphs of three algebraic functions.

Trigonometric Functions We review trigonometric functions in Section 1.6. The graphs of the sine and cosine functions are shown in Figure 10.

Exponential Functions Functions of the form $f(x) = a^x$, where the base $a > 0$ is a positive constant and $a \neq 1$, are called **exponential functions**. All exponential functions have domain $(-\infty, \infty)$and range $(0, \infty)$. So an exponential function never assumes the value 0. The graphs of some exponential functions are shown in Figure 11.

<u>Logarithmic Functions</u> These are the functions $f(x) = \log_a x$ where the base $a \neq 1$ is a positive constant. They are the *inverse functions* of the exponential functions as shown in Figure 12.

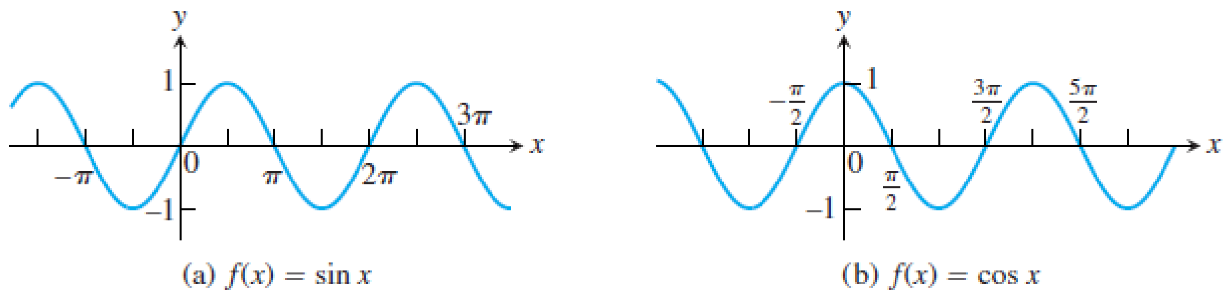

FIGURE 10: Graphs of the sine and cosin functions.

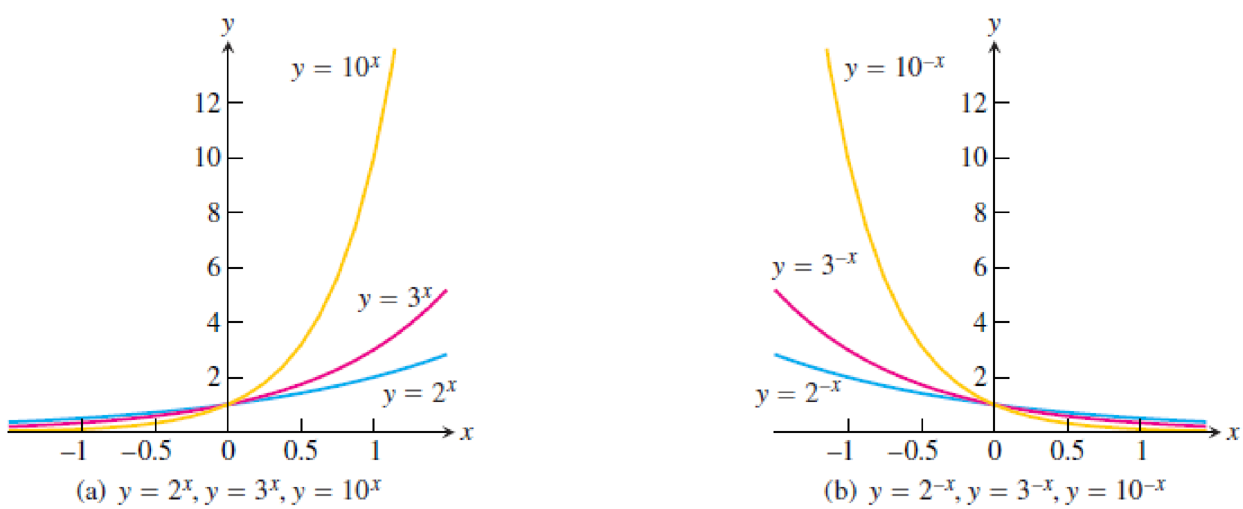

FIGURE 11: Graphs of exponential functions.

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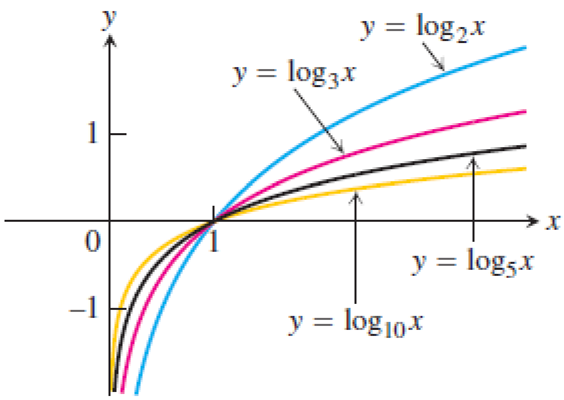
Transcendental Functions These are functions that are not algebraic. They include the trigonometric, inverse trigonometric, exponential, and logarithmic functions, and many other functions. Its graph takes the shape of a cable, like a telephone line or TV cable see Figure 13.



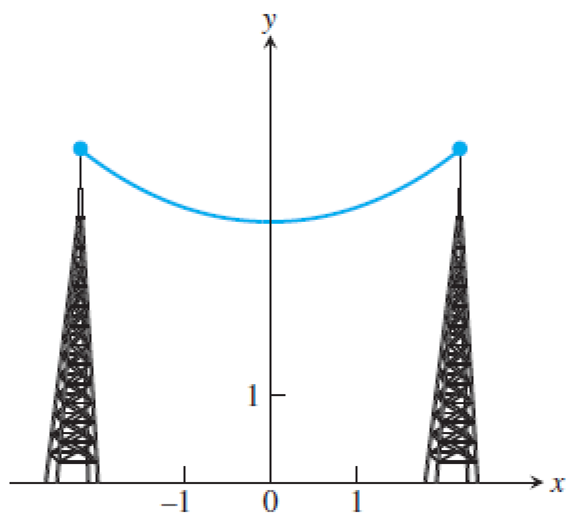


FIGURE 12: Graphs of logarithmic function. FIGURE 13: Graph of transcendental function

Even Functions and Odd Functions: Symmetry

The graphs of *even* and *odd* functions have characteristic symmetry properties.

DEFINITIONS Even Function, Odd Function

A function $y = f(x)$ is an

$$\textbf{even function of } x \quad \text{if } f(-x) = f(x),$$
$$\textbf{odd function of } x \quad \text{if } f(-x) = -f(x),$$

for every x in the function's domain.

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The names even and odd come from powers of x. If y is an even power of x, as in $y = x^2$ or $y = x^4$, it is an even function of x (because $(-x)^2 = x^2$ and $(-x)^4 = x^4$). If y is an odd power of x, as in $y = x$ or $y = x^3$, it is an odd function of x (because $(-x) = -x$ and $(-x)^3 = -x^3$)

The graph of an even function is **symmetric about the *y*-axis**. Since $f(-x) = f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, y)$ lies on the graph

The graph of an odd function is **symmetric about the origin**. Since $f(-x) = -f(x)$, a point (x, y) lies on the graph if and only if the point $(-x, -y)$ lies on the graph as shown in Figure 14.

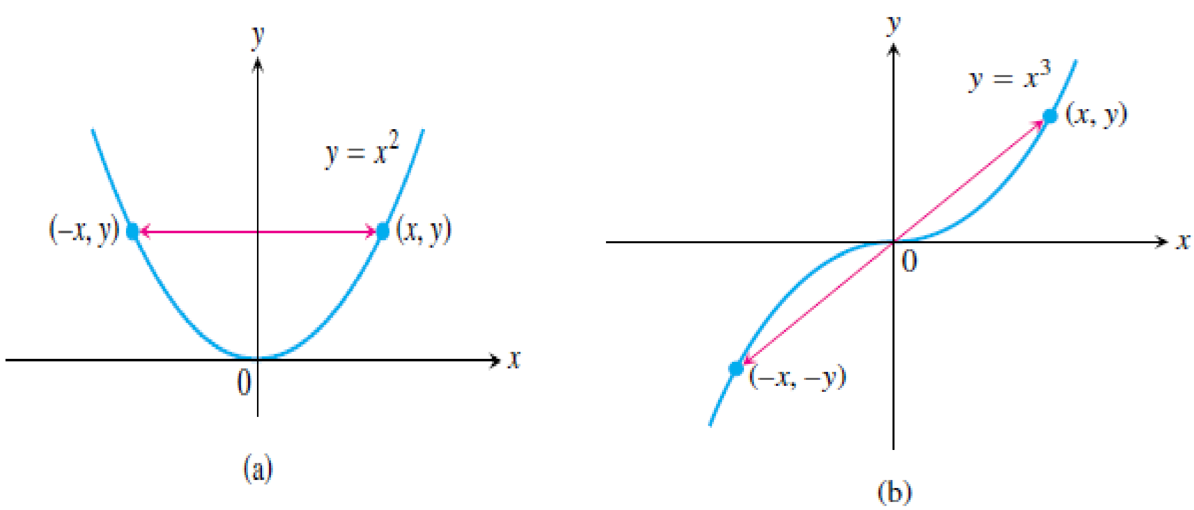

FIGURE 14: In part (a) the graph of $y = x^2$ (an even function) is symmetric about the y-axis. The graph of $y = x^3$ (an odd function) in part (b) is symmetric about the origin.

H.W

In Exercises 1– 4, identify each function as a constant function, linear function, power function, polynomial (state its degree), rational func- tion, algebraic function, trigonometric function, exponential function, or logarithmic function. Remember that some functions can fall into more than one category.

1. a. $f(x) = 7 - 3x$ b. $g(x) = \sqrt[5]{x}$
   c. $h(x) = \dfrac{x^2 - 1}{x^2 + 1}$ d. $r(x) = 8^x$
2. a. $F(t) = t^4 - t$ b. $G(t) = 5^t$
   c. $H(z) = \sqrt{z^3 + 1}$ d. $R(z) = \sqrt[3]{z^7}$
3. a. $y = \dfrac{3 + 2x}{x - 1}$ b. $y = x^{5/2} - 2x + 1$
   c. $y = \tan \pi x$ d. $y = \log_7 x$
4. a. $y = \log_5 \left(\dfrac{1}{t}\right)$ b. $f(z) = \dfrac{z^5}{\sqrt{z} + 1}$
   c. $g(x) = 2^{1/x}$ d. $w = 5 \cos \left(\dfrac{t}{2} + \dfrac{\pi}{6}\right)$

Trigonometric Functions

This section reviews the basic trigonometric functions. The trigonometric functions are important because they are periodic, or repeating, and therefore model many naturally occurring periodic processes.

In navigation and astronomy, angles are measured in degrees, but in calculus it is best to use units called *radians* because of the way they simplify later calculations

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The **radian measure** of the angle ACB at the center of the unit circle (Figure 15) equals the length of the arc that ACB cuts from the unit circle. Figure 1.63 shows that $s = r\theta$ is the **length of arc** cut from a circle of radius r when the subtending angle θ producing the arc is measured in radians.

Since the circumference of the circle is 2π and one complete revolution of a circle is 360°, the relation between radians and degrees is given by

$$\pi\ radius = 180^{\circ}$$

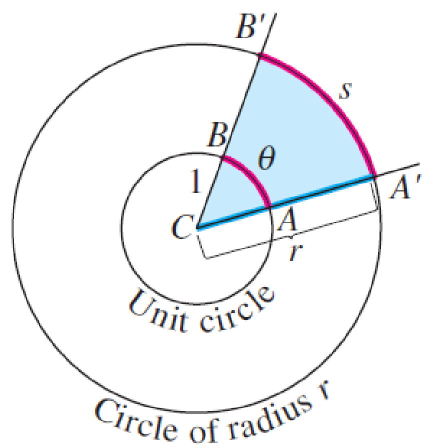


FIGURE 15 The radian measure of angle ACB is the length θ of arc AB on the unit circle centered at C.

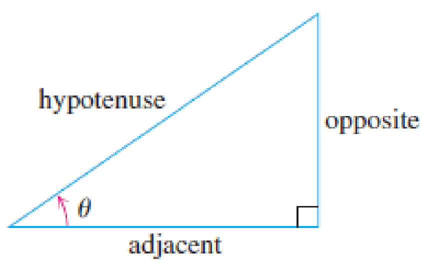
The Six Basic Trigonometric Functions

You are probably familiar with defining the trigonometric functions of an acute angle in terms of the sides of a right triangle (Figure 16).

$$\textbf{sine:}\quad \sin\theta = \frac{y}{r} \qquad \textbf{cosecant:}\quad \csc\theta = \frac{r}{y}$$

$$\textbf{cosine:}\quad \cos\theta = \frac{x}{r} \qquad \textbf{secant:}\quad \sec\theta = \frac{r}{x}$$

$$\textbf{tangent:}\quad \tan\theta = \frac{y}{x} \qquad \textbf{cotangent:}\quad \cot\theta = \frac{x}{y}$$



$$\sin\theta = \frac{\text{opp}}{\text{hyp}} \qquad \csc\theta = \frac{\text{hyp}}{\text{opp}}$$

$$\cos\theta = \frac{\text{adj}}{\text{hyp}} \qquad \sec\theta = \frac{\text{hyp}}{\text{adj}}$$

$$\tan\theta = \frac{\text{opp}}{\text{adj}} \qquad \cot\theta = \frac{\text{adj}}{\text{opp}}$$

FIGURE 16 Trigonometric ratios of an acute angle.

Notice also the following definitions, whenever the quotients are defined$\bar{y}$

$$\tan\theta = \frac{\sin\theta}{\cos\theta} \qquad \cot\theta = \frac{1}{\tan\theta}$$

$$\sec\theta = \frac{1}{\cos\theta} \qquad \csc\theta = \frac{1}{\sin\theta}$$

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Most calculators and computers readily provide values of the trigonometric functions for angles given in either radians or degrees.

TABLE 1.4 Values of $\sin\theta$, $\cos\theta$, and $\tan\theta$ for selected values of θ

Degrees	−180	−135	−90	−45	0	30	45	60	90	120	135	150	180	270	360
θ (radians)	$-\pi$	$\frac{-3\pi}{4}$	$\frac{-\pi}{2}$	$\frac{-\pi}{4}$	0	$\frac{\pi}{6}$	$\frac{\pi}{4}$	$\frac{\pi}{3}$	$\frac{\pi}{2}$	$\frac{2\pi}{3}$	$\frac{3\pi}{4}$	$\frac{5\pi}{6}$	π	$\frac{3\pi}{2}$	2π
$\sin\theta$	0	$\frac{-\sqrt{2}}{2}$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{1}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{3}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	−1	0
$\cos\theta$	−1	$\frac{-\sqrt{2}}{2}$	0	$\frac{\sqrt{2}}{2}$	1	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{2}}{2}$	$\frac{1}{2}$	0	$-\frac{1}{2}$	$\frac{-\sqrt{2}}{2}$	$\frac{-\sqrt{3}}{2}$	−1	0	1
$\tan\theta$	0	1		−1	0	$\frac{\sqrt{3}}{3}$	1	$\sqrt{3}$		$-\sqrt{3}$	−1	$\frac{-\sqrt{3}}{3}$	0		0

When we graph trigonometric functions in the coordinate plane, we usually denote the in- dependent variable by x instead of θ. See Figure 17.

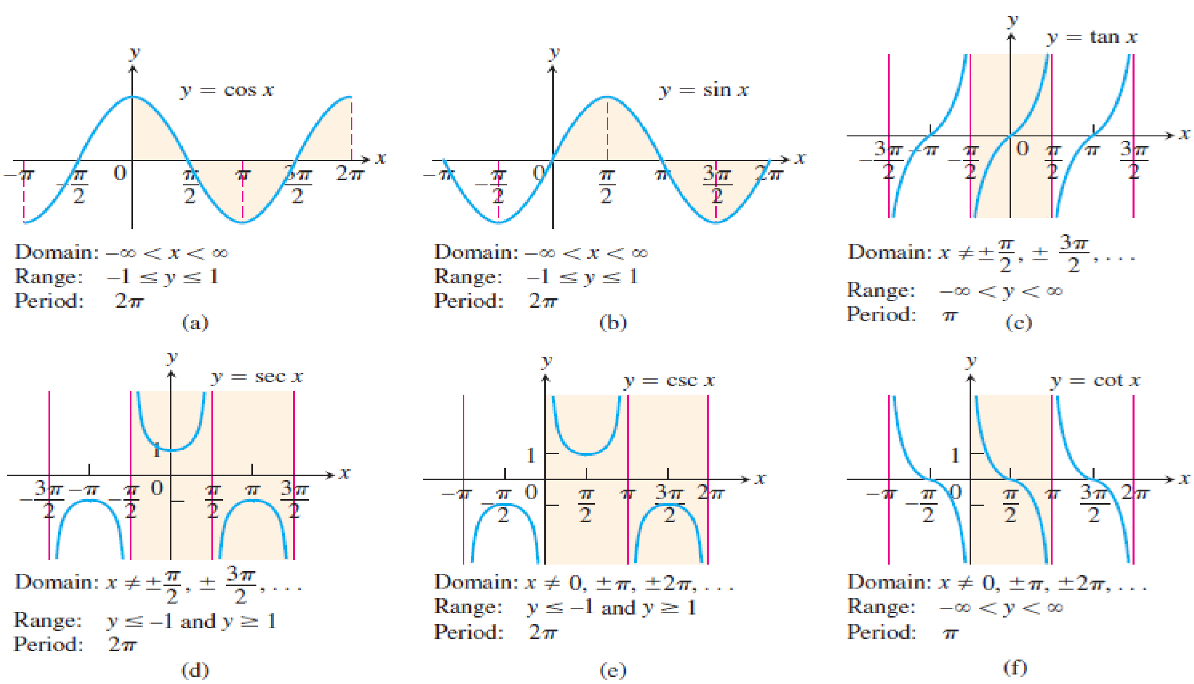

FIGURE 17 Graphs of the (a) cosine, (b) sine, (c) tangent, (d) secant, (e) cosecant, and (f) cotangent functions using radian measure. The shading for each trigonometric function indicates its periodicity.

The symmetries in the graphs in Figure 17 reveal that the cosine and secant functions are even and the other four fuctions are odd:

Even	Odd
$\cos(-x) = \cos x$	$\sin(-x) = -\sin x$
$\sec(-x) = \sec x$	$\tan(-x) = -\tan x$
	$\csc(-x) = -\csc x$
	$\cot(-x) = -\cot x$

Identities

The coordinates of any point $P(x, y)$ in the plane can be expressed in terms of the point's distance from the origin and the angle that ray OP makes with the positive x-axis (Figure 18). Since $x = r\,cos\theta$ and $y = r\,sin\theta$, we have

When $r = 1$ we can apply the Pythagorean theorem to the reference right triangle in Figure 18 and obtain the equation

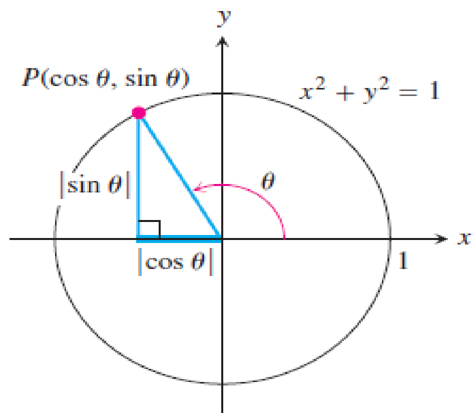

FIGURE 18 The reference triangle for a general angle θ.

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$$\cos^2\theta + \sin^2\theta = 1. \tag{1}$$

This equation, true for all values of θ, is the most frequently used identity in trigonometry. Dividing this identity in turn by $\cos^2\theta$ and $\sin^2\theta$ gives

$$1 + \tan^2\theta = \sec^2\theta.$$
$$1 + \cot^2\theta = \csc^2\theta.$$

Double-Angle Formulas

$$\begin{aligned} \cos 2\theta &= \cos^2\theta - \sin^2\theta \\ \sin 2\theta &= 2\sin\theta\cos\theta \end{aligned} \tag{3}$$

Half-Angle Formulas

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2} \tag{4}$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2} \tag{5}$$

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