This section introduces a few rules that allow us to differentiate a great variety of functions. By proving these rules here, we can differentiate functions without having to apply the definition of the derivative each time.

Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is zero.

RULE 1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

EXAMPLE 1

If f has the constant value $f(x) = 8$, then

$$\frac{df}{dx} = \frac{d}{dx}(8) = 0.$$

Similarly,

$$\frac{d}{dx}\left(-\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \frac{d}{dx}\left(\sqrt{3}\right) = 0.$$

Proof of Rule 1 We apply the definition of derivative to $f(x) = c$, the function whose outputs have the constant value c (Figure 3.8). At every value of x, we find that

$$f'(x) = \lim_{h\to 0}\frac{f(x+h) - f(x)}{h} = \lim_{h\to 0}\frac{c - c}{h} = \lim_{h\to 0} 0 = 0.$$

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The second rule tells how to differentiate x^n if n is a positive integer.

RULE 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx}x^n = nx^{n-1}.$$

Second Proof of Rule 2 If $f(x) = x^n$, then $f(x+h) = (x+h)^n$. Since n is a positive integer, we can expand $(x+h)^n$ by the Binomial Theorem to get

$$\begin{aligned} f'(x) &= \lim_{h\to 0}\frac{f(x+h)-f(x)}{h} = \lim_{h\to 0}\frac{(x+h)^n - x^n}{h} \\ &= \lim_{h\to 0}\frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n\right] - x^n}{h} \\ &= \lim_{h\to 0}\frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \cdots + nxh^{n-1} + h^n}{h} \\ &= \lim_{h\to 0}\left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \cdots + nxh^{n-2} + h^{n-1}\right] \\ &= nx^{n-1} \end{aligned}$$

■

The third rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

RULE 3 Constant Multiple Rule

If u is a differentiable function of x, and c is a constant, then

$$\frac{d}{dx}(cu) = c\frac{du}{dx}.$$

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EXAMPLE 3

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

Proof of Rule 3

$$\frac{d}{dx}cu = \lim_{h \to 0} \frac{cu(x+h) - cu(x)}{h} \qquad \text{Derivative definition with } f(x) = cu(x)$$

$$= c \lim_{h \to 0} \frac{u(x+h) - u(x)}{h} \qquad \text{Limit property}$$

$$= c\frac{du}{dx} \qquad u \text{ is differentiable.}$$

■

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

RULE 4 Derivative Sum Rule

If u and v are differentiable functions of x, then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

Email: firas.thair.almaliky@uomus.edu.iq

EXAMPLE 4 Derivative of a Sum

$$y = x^4 + 12x$$
$$\frac{dy}{dx} = \frac{d}{dx}(x^4) + \frac{d}{dx}(12x)$$
$$= 4x^3 + 12$$

Proof of Rule 4 We apply the definition of derivative to $f(x) = u(x) + v(x)$:

$$\frac{d}{dx}[u(x) + v(x)] = \lim_{h\to 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h}$$
$$= \lim_{h\to 0}\left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h}\right]$$
$$= \lim_{h\to 0}\frac{u(x+h) - u(x)}{h} + \lim_{h\to 0}\frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}.$$

EXAMPLE 5 Derivative of a Polynomial

$$y = x^3 + \frac{4}{3}x^2 - 5x + 1$$
$$\frac{dy}{dx} = \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1)$$
$$= 3x^2 + \frac{4}{3}\cdot 2x - 5 + 0$$
$$= 3x^2 + \frac{8}{3}x - 5$$

Products and Quotients

The derivative of a product of two functions is the sum of *two* products, as we now explain.

RULE 5 Derivative Product Rule

If u and v are differentiable at x, then so is their product uv, and

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$

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Proof of Rule 5

$$\frac{d}{dx}(uv) = \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x)v(x)}{h}$$

To change this fraction into an equivalent one that contains difference quotients for the derivatives of u and v, we subtract and add $u(x+h)v(x)$ in the numerator:

$$\begin{aligned}\frac{d}{dx}(uv) &= \lim_{h\to 0} \frac{u(x+h)v(x+h) - u(x+h)v(x) + u(x+h)v(x) - u(x)v(x)}{h}\\ &= \lim_{h\to 0}\left[u(x+h)\frac{v(x+h)-v(x)}{h} + v(x)\frac{u(x+h)-u(x)}{h}\right]\\ &= \lim_{h\to 0} u(x+h)\cdot \lim_{h\to 0}\frac{v(x+h)-v(x)}{h} + v(x)\cdot \lim_{h\to 0}\frac{u(x+h)-u(x)}{h}.\end{aligned}$$

As h approaches zero, $u(x+h)$ approaches $u(x)$ because u, being differentiable at x, is continuous at x. The two fractions approach the values of dv/dx at x and du/dx at x. In short,

$$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}.$$ ■

EXAMPLE 7 Using the Product Rule

Find the derivative of

$$y = \frac{1}{x}\left(x^2 + \frac{1}{x}\right).$$

Solution We apply the Product Rule with $u = 1/x$ and $v = x^2 + (1/x)$:

$$\begin{aligned}\frac{d}{dx}\left[\frac{1}{x}\left(x^2+\frac{1}{x}\right)\right] &= \frac{1}{x}\left(2x - \frac{1}{x^2}\right) + \left(x^2+\frac{1}{x}\right)\left(-\frac{1}{x^2}\right)\\ &= 2 - \frac{1}{x^3} - 1 - \frac{1}{x^3}\\ &= 1 - \frac{2}{x^3}.\end{aligned}$$

$\frac{d}{dx}(uv) = u\frac{dv}{dx} + v\frac{du}{dx}$, and $\frac{d}{dx}\left(\frac{1}{x}\right) = -\frac{1}{x^2}$ by Example 3, Section 2.7.

■

Email: firas.thair.almaliky@uomus.edu.iq

EXAMPLE 8 Derivative from Numerical Values

Let $y = uv$ be the product of the functions u and v. Find $y'(2)$ if

$$u(2) = 3, \quad u'(2) = -4, \quad v(2) = 1, \quad \text{and} \quad v'(2) = 2.$$

Solution From the Product Rule, in the form

$$y' = (uv)' = uv' + vu',$$

EXAMPLE 9 Differentiating a Product in Two Ways

Find the derivative of $y = (x^2 + 1)(x^3 + 3)$.

Solution

(a) From the Product Rule with $u = x^2 + 1$ and $v = x^3 + 3$, we find

$$\begin{aligned} \frac{d}{dx}\left[\left(x^2 + 1\right)\left(x^3 + 3\right)\right] &= (x^2 + 1)(3x^2) + (x^3 + 3)(2x) \\ &= 3x^4 + 3x^2 + 2x^4 + 6x \\ &= 5x^4 + 3x^2 + 6x. \end{aligned}$$

RULE 6 Derivative Quotient Rule

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient u/v is differentiable at x, and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}.$$

EXAMPLE 10 Using the Quotient Rule

Find the derivative of

$$y = \frac{t^2 - 1}{t^2 + 1}.$$

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Solution

We apply the Quotient Rule with $u = t^2 - 1$ and $v = t^2 + 1$:

$$\frac{dy}{dt} = \frac{(t^2+1)\cdot 2t - (t^2-1)\cdot 2t}{(t^2+1)^2} \qquad \frac{d}{dt}\left(\frac{u}{v}\right) = \frac{v(du/dt) - u(dv/dt)}{v^2}$$

$$= \frac{2t^3 + 2t - 2t^3 + 2t}{(t^2+1)^2}$$

$$= \frac{4t}{(t^2+1)^2}.$$

Proof of Rule 6

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \lim_{h\to 0} \frac{\dfrac{u(x+h)}{v(x+h)} - \dfrac{u(x)}{v(x)}}{h}$$

$$= \lim_{h\to 0} \frac{v(x)u(x+h) - u(x)v(x+h)}{hv(x+h)v(x)}$$

Negative Integer Powers of *x*

The Power Rule for negative integers is the same as the rule for positive integers.

RULE 7 Power Rule for Negative Integers

If n is a negative integer and $x \neq 0$, then

$$\frac{d}{dx}(x^n) = nx^{n-1}.$$

EXAMPLE 11

(a) $\dfrac{d}{dx}\left(\dfrac{1}{x}\right) = \dfrac{d}{dx}(x^{-1}) = (-1)x^{-2} = -\dfrac{1}{x^2}$ — Agrees with Example 3, Section 2.7

(b) $\dfrac{d}{dx}\left(\dfrac{4}{x^3}\right) = 4\dfrac{d}{dx}(x^{-3}) = 4(-3)x^{-4} = -\dfrac{12}{x^4}$

Second- and Higher-Order Derivatives

If $y = f(x)$ is a differentiable function, then its derivative $f'(x)$ is also a function. If f' is also differentiable, then we can differentiate f' to get a new function of x denoted by f''. So $f'' = (f')'$. The function f'' is called the **second derivative** of f because it is the derivative of the first derivative. Notationally,

$$f''(x) = \frac{d^2y}{dx^2} = \frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{dy'}{dx} = y'' = D^2(f)(x) = D_x^2 f(x).$$

EXAMPLE 12 Finding Higher Derivatives

The first four derivatives of $y = x^3 - 3x^2 + 2$ are

First derivative: $y' = 3x^2 - 6x$

Second derivative: $y'' = 6x - 6$

Third derivative: $y''' = 6$

Fourth derivative: $y^{(4)} = 0$.

The function has derivatives of all orders, the fifth and later derivatives all being zero. ■

EX- Find $\frac{dy}{dx}$ for the following functions :

a) $y = (x^2+1)^5$

b) $y = [(5-x)(4-2x)]^2$

c) $y = (2x^3 - 3x^2 + 6x)^{-5}$

d) $y = \frac{12}{x} - \frac{4}{x^3} + \frac{3}{x^4}$

e) $y = \frac{(x^2+x)(x^2-x+1)}{x^3}$

f) $y = \frac{x^2-1}{x^2+x-2}$

Sol.-

a) $\frac{dy}{dx} = 5(x^2+1)^4 . 2x = 10x(x^2+1)^4$

b) $\frac{dy}{dx} = 2[(5-x)(4-2x)][-2(5-x)-(4-2x)]$

$= 8(5-x)(2-x)(2x-7)$

c) $\frac{dy}{dx} = -5(2x^3 - 3x^2 + 6x)^{-6}(6x^2 - 6x + 6)$

$= -30(2x^3 - 3x^2 + 6x)^{-6}(x^2 - x + 1)$

Email: firas.thair.almaliky@uomus.edu.iq

d) $y = 12x^{-1} - 4x^{-3} + 3x^{-4} \Rightarrow \frac{dy}{dx} = -12x^{-2} + 12x^{-4} - 12x^{-5}$

$$\Rightarrow \frac{dy}{dx} = -\frac{12}{x^2} + \frac{12}{x^4} - \frac{12}{x^5}$$

e) $y = \frac{(x+1)(x^2-x+1)}{x^3} \Rightarrow$

$$\frac{dy}{dx} = \frac{x^3\left[(x^2-x+1)+(x+1)(2x-1)\right] - 3x^2(x+1)(x^2-x+1)}{x^6} = -\frac{3}{x^4}$$

f) $\frac{dy}{dx} = \frac{2x(x^2+x-2)-(x^2-1)(2x+1)}{(x^2+x-2)^2} = \frac{x^2-2x+1}{(x^2+x-2)^2}$

Derivatives of Trigonometric Functions

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

EXAMPLE 1 Derivatives Involving the Sine

(a) $y = x^2 - \sin x$:

$$\frac{dy}{dx} = 2x - \frac{d}{dx}(\sin x) \qquad \text{Difference Rule}$$
$$= 2x - \cos x.$$

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(b) $y = x^2 \sin x$:

$$\frac{dy}{dx} = x^2 \frac{d}{dx}\left(\sin x\right) + 2x \sin x \qquad \text{Product Rule}$$
$$= x^2 \cos x + 2x \sin x.$$

(c) $y = \dfrac{\sin x}{x}$:

$$\frac{dy}{dx} = \frac{x \cdot \frac{d}{dx}\left(\sin x\right) - \sin x \cdot 1}{x^2} \qquad \text{Quotient Rule}$$
$$= \frac{x \cos x - \sin x}{x^2}.$$

■

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x$$

EXAMPLE 2 Derivatives Involving the Cosine

(a) $y = 5x + \cos x$:

$$\frac{dy}{dx} = \frac{d}{dx}(5x) + \frac{d}{dx}\left(\cos x\right) \qquad \text{Sum Rule}$$
$$= 5 - \sin x.$$

(b) $y = \sin x \cos x$:

$$\frac{dy}{dx} = \sin x \frac{d}{dx}\left(\cos x\right) + \cos x \frac{d}{dx}\left(\sin x\right) \qquad \text{Product Rule}$$
$$= \sin x(-\sin x) + \cos x(\cos x)$$
$$= \cos^2 x - \sin^2 x.$$

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(c) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} && \text{Quotient Rule} \\
&= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2} \\
&= \frac{1 - \sin x}{(1 - \sin x)^2} && \sin^2 x + \cos^2 x = 1 \\
&= \frac{1}{1 - \sin x}.
\end{aligned}$$

■

Derivatives of the Other Trigonometric Functions

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

EXAMPLE 3

Find $d(\tan x)/dx$.

Solution

$$\begin{aligned}
\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) &= \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} && \text{Quotient Rule} \\
&= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x} \\
&= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} \\
&= \frac{1}{\cos^2 x} = \sec^2 x
\end{aligned}$$

■

Email: firas.thair.almaliky@uomus.edu.iq

EXAMPLE 4

Find y'' if $y = \sec x$.

Solution

$$\begin{aligned}
y &= \sec x \\
y' &= \sec x \tan x \\
y'' &= \frac{d}{dx}(\sec x \tan x) \\
&= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \qquad \text{Product Rule} \\
&= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \\
&= \sec^3 x + \sec x \tan^2 x
\end{aligned}$$

■

EX- **Find $\frac{dy}{dx}$ for the following functions :**

a) $y = \tan(3x^2)$ b) $y = (\csc x + \cot x)^2$

c) $y = 2\sin\frac{x}{2} - x\cos\frac{x}{2}$ d) $y = \tan^2(\cos x)$

e) $x + \tan(xy) = 0$ f) $y = \sec^4 x - \tan^4 x$

Sol.-

a) $\frac{dy}{dx} = \sec^2(3x^2).6x = 6x.\sec^2(3x^2)$

b) $\frac{dy}{dx} = 2(\csc x + \cot x)(-\csc x.\cot x - \csc^2 x) = -2\csc x.(\csc x + \cot x)^2$

c) $\frac{dy}{dx} = 2\cos\frac{x}{2}.\frac{1}{2} - \left[x(-\sin\frac{x}{2}).\frac{1}{2} + \cos\frac{x}{2}\right] = \frac{x}{2}.\sin\frac{x}{2}$

d) $\frac{dy}{dx} = 2.\tan(\cos x).\sec^2(\cos x).(-\sin x) = -2.\sin x.\tan(\cos x).\sec^2(\cos x)$

e) $1 + \sec^2(xy).(x\frac{dy}{dx} + y) = 0 \Rightarrow \frac{dy}{dx} = -\frac{1 + y.\sec^2(xy)}{x.\sec^2(xy)} = -\frac{\cos^2(xy) + y}{x}$

f) $\frac{dy}{dx} = 4\sec^3 x.\sec x.\tan x - 4.\tan^3 x.\sec^2 x = 4\tan x.\sec^2 x$

Email: firas.thair.almaliky@uomus.edu.iq

The Chain Rule and Parametric Equations

THEOREM 3 The Chain Rule

If $f(u)$ is differentiable at the point $u = g(x)$ and $g(x)$ is differentiable at x, then the composite function $(f \circ g)(x) = f(g(x))$ is differentiable at x, and

$$(f \circ g)'(x) = f'(g(x)) \cdot g'(x).$$

In Leibniz's notation, if $y = f(u)$ and $u = g(x)$, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx},$$

where dy/du is evaluated at $u = g(x)$.

EXAMPLE 1 Applying the Chain Rule

An object moves along the x-axis so that its position at any time $t \geq 0$ is given by $x(t) = \cos(t^2 + 1)$. Find the velocity of the object as a function of t.

Solution We know that the velocity is dx/dt. In this instance, x is a composite function: $x = \cos(u)$ and $u = t^2 + 1$. We have

$$\frac{dx}{du} = -\sin(u) \qquad x = \cos(u)$$

$$\frac{du}{dt} = 2t. \qquad u = t^2 + 1$$

By the Chain Rule,

$$\begin{aligned} \frac{dx}{dt} &= \frac{dx}{du} \cdot \frac{du}{dt} \\ &= -\sin(u) \cdot 2t \qquad \frac{dx}{du} \text{ evaluated at } u \\ &= -\sin(t^2 + 1) \cdot 2t \\ &= -2t \sin(t^2 + 1). \end{aligned}$$

■

Email: firas.thair.almaliky@uomus.edu.iq

Parametric Formula for dy/dx

If all three derivatives exist and $dx/dt \neq 0$,

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}. \qquad (2)$$

EXAMPLE 2 Differentiating with a Parameter

If $x = 2t + 3$ and $y = t^2 - 1$, find the value of dy/dx at $t = 6$.

Solution Equation (2) gives dy/dx as a function of t:

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{2t}{2} = t = \frac{x-3}{2}.$$

EX-3 – **Use the chain rule to express dy/dx in terms of x and y:**

a) $y = \dfrac{t^2}{t^2+1}$ and $t = \sqrt{2x+1}$

b) $y = \dfrac{1}{t^2+1}$ and $x = \sqrt{4t+1}$

c) $y = \left(\dfrac{t-1}{t+1}\right)^2$ and $x = \dfrac{1}{t^2} - 1$ at $t = 2$

d) $y = 1 - \dfrac{1}{t}$ and $t = \dfrac{1}{1-x}$ at $x = 2$

Sol.-

a) $y = \dfrac{t^2}{t^2+1} \Rightarrow \dfrac{dy}{dt} = \dfrac{2t(t^2+1) - 2t.t^2}{(t^2+1)^2} = \dfrac{2t}{(t^2+1)^2}$

$t = (2x+1)^{\frac{1}{2}} \Rightarrow \dfrac{dt}{dx} = \dfrac{1}{2}.(2x+1)^{-\frac{1}{2}}.2 = \dfrac{1}{\sqrt{2x+1}}$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{2t}{(t^2+1)^2}\cdot\frac{1}{\sqrt{2x+1}} = \frac{2\sqrt{2x+1}}{((2x+1)+1)^2}\cdot\frac{1}{\sqrt{2x+1}} = \frac{1}{2(x+1)^2}$$

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$$b)\quad y=(t^2+1)^{-1}\Rightarrow \frac{dy}{dx}=-2t(t^2+1)^{-2}=-\frac{2t}{(t^2+1)^2}$$

$$x=(4t+1)^{\frac{1}{2}}\Rightarrow \frac{dx}{dt}=\frac{1}{2}(4t+1)^{-\frac{1}{2}}.4=\frac{2}{\sqrt{4t+1}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-\frac{2t}{(t^2+1)^2}\div\frac{2}{\sqrt{4t+1}}=-\frac{t\sqrt{4t+1}}{(t^2+1)^2}$$

$$=-\frac{x^2-1}{4}.x\div\frac{1}{y^2}=-\frac{xy^2(x^2-1)}{4}$$

$$where\quad x=\sqrt{4t+1}\Rightarrow t=\frac{x^2-1}{4}$$

$$where\quad y=\frac{1}{t^2+1}\Rightarrow t^2+1=\frac{1}{y}$$

$$c)\quad y=\left(\frac{t-1}{t+1}\right)^2\Rightarrow \frac{dy}{dt}=2\left(\frac{t-1}{t+1}\right)\frac{t+1-(t-1)}{(t+1)^2}=\frac{4(t-1)}{(t+1)^3}$$

$$\Rightarrow\left[\frac{dy}{dt}\right]_{t=2}=\frac{4(2-1)}{(2+1)^3}=\frac{4}{27}$$

$$x=\frac{1}{t^2}-1\Rightarrow\frac{dx}{dt}=-\frac{2}{t^3}\Rightarrow\left[\frac{dx}{dt}\right]_{t=2}=-\frac{2}{2^3}=-\frac{1}{4}$$

$$\left[\frac{dy}{dx}\right]_{t=2}=\left[\frac{dy}{dt}\div\frac{dx}{dt}\right]_{t=2}=\frac{4}{27}\div\left(-\frac{1}{4}\right)=-\frac{16}{27}$$

$$d)\quad t=\frac{1}{1-x}=\frac{1}{1-2}=-1\qquad at\quad x=2$$

$$y=1-\frac{1}{t}\Rightarrow\frac{dy}{dt}=\frac{1}{t^2}\Rightarrow\left[\frac{dy}{dt}\right]_{t=-1}=\frac{1}{(-1)^2}=1$$

$$t=(1-x)^{-1}\Rightarrow\frac{dt}{dx}=-(1-x)^{-2}(-1)=\frac{1}{(1-x)^2}$$

$$\Rightarrow\left[\frac{dt}{dx}\right]_{x=2}=\frac{1}{(1-2)^2}=1$$

$$\left[\frac{dy}{dx}\right]_{x=2}=\left[\frac{dy}{dt}\right]_{x=2}.\left[\frac{dt}{dx}\right]_{x=2}=1*1=1$$

Implicit Differentiation

Implicit Differentiation

1. Differentiate both sides of the equation with respect to x, treating y as a differentiable function of x.
2. Collect the terms with dy/dx on one side of the equation.
3. Solve for dy/dx.

EXAMPLE 1 Differentiating Implicitly

Find dy/dx if $y^2 = x^2 + \sin xy$

Solution

$$y^2 = x^2 + \sin xy$$

$$\frac{d}{dx}\left(y^2\right) = \frac{d}{dx}\left(x^2\right) + \frac{d}{dx}\left(\sin xy\right)$$ Differentiate both sides with respect to x . . .

$$2y\frac{dy}{dx} = 2x + (\cos xy)\frac{d}{dx}(xy)$$ . . . treating y as a function of x and using the Chain Rule.

$$2y\frac{dy}{dx} = 2x + (\cos xy)\left(y + x\frac{dy}{dx}\right)$$ Treat xy as a product.

$$2y\frac{dy}{dx} - (\cos xy)\left(x\frac{dy}{dx}\right) = 2x + (\cos xy)y$$ Collect terms with dy/dx . . .

$$(2y - x\cos xy)\frac{dy}{dx} = 2x + y\cos xy$$ . . . and factor out dy/dx.

$$\frac{dy}{dx} = \frac{2x + y\cos xy}{2y - x\cos xy}$$ Solve for dy/dx by dividing.

<u>EX-6</u>- Find $\frac{dy}{dx}$ for the following functions:

a) $x^2 . y^2 = x^2 + y^2$

b) $(x+y)^3 + (x-y)^3 = x^4 + y^4$

c) $\frac{x-y}{x-2y} = 2$ at $P(3,1)$

d) $xy + 2x - 5y = 2$ at $P(3,2)$

<u>Sol.</u>

$$a)\ x^2(2y\frac{dy}{dx}) + y^2(2x) = 2x + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2y - y}$$

$$b)\ 3(x+y)^2(1+\frac{dy}{dx}) + 3(x-y)^2(1-\frac{dy}{dx}) = 4x^3 + 4y^3\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 3(x+y)^2 - 3(x-y)^2}{3(x+y)^2 - 3(x-y)^2 - 4y^3} \Rightarrow \frac{dy}{dx} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

$$c)\ \frac{(x-2y)(1-\frac{dy}{dx}) - (x-y)(1-2\frac{dy}{dx})}{(x-2y)^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,1)} = \frac{1}{3}$$

$$d)\ x\frac{dy}{dx} + y + 2 - 5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{5-x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,2)} = \frac{2+2}{5-3} = 2$$

The Exponential Function

If u is any differentiable function of x, then

$$\frac{d}{dx}e^u = e^u\frac{du}{dx}.$$

If $a > 0$ and u is a differentiable function of x, then a^u is a differentiable function of x and

$$\frac{d}{dx}a^u = a^u \ln a\,\frac{du}{dx}.$$

Email: firas.thair.almaliky@uomus.edu.iq

EX- –Find $\dfrac{dy}{dx}$ for the following functions :

a) $y = 2^{3x}$ b) $y = 2^x.3^x$

c) $y = (2^x)^2$ d) $y = x.2^{x^2}$

e) $y = e^{(x+e^{5x})}$ f) $y = e^{\sqrt{1+5x^2}}$

Sol.-

$$a)\ y = 2^{3x} \Rightarrow \frac{dy}{dx} = 2^{3x} * 3\ln 2$$

$$b)\ y = 2^x.3^x \Rightarrow y = 6^x \Rightarrow \frac{dy}{dx} = 6^x.\ln 6$$

$$c)\ y = (2^x)^2 \Rightarrow y = 2^{2x} \Rightarrow \frac{dy}{dx} = 2^{2x}\ln 2.2 = 2^{2x+1}\ln 2$$

$$d)\ y = x.2^{x^2} \Rightarrow \frac{dy}{dx} = x.2^{x^2}\ln 2.2x + 2^{x^2} = 2^{x^2}(2x^2\ln 2 + 1)$$

$$e)\ y = e^{(x+e^{5x})} \Rightarrow \frac{dy}{dx} = e^{(x+e^{5x})}(1+5e^{5x})$$

$$f)\ y = e^{(1+5x^2)^{\frac{1}{2}}} \Rightarrow \frac{dy}{dx} = e^{(1+5x^2)^{\frac{1}{2}}}\frac{1}{2}(1+5x^2)^{-\frac{1}{2}}.10x = e^{\sqrt{1+5x^2}}\frac{5x}{\sqrt{1+5x^2}}$$

Natural Logarithms

$$\frac{d}{dx}\ln u = \frac{1}{u}\frac{du}{dx}, \qquad u > 0$$

Email: firas.thair.almaliky@uomus.edu.iq

THEOREM Properties of Logarithms

For any numbers $a > 0$ and $x > 0$, the natural logarithm satisfies the following rules:

1. *Product Rule*: $\ln ax = \ln a + \ln x$
2. *Quotient Rule*: $\ln \frac{a}{x} = \ln a - \ln x$
3. *Reciprocal Rule*: $\ln \frac{1}{x} = -\ln x$ Rule 2 with $a = 1$
4. *Power Rule*: $\ln x^r = r \ln x$ r rational

Logarithms with Base a

$$\frac{d}{dx}(\log_a u) = \frac{1}{\ln a} \cdot \frac{1}{u} \frac{du}{dx}$$

EX- **– Find $\frac{dy}{dx}$ for the following functions :**

a) $y = \log_{10} e^x$

b) $y = \log_5 (x+1)^2$

c) $y = \log_2 (3x^2+1)^3$

d) $y = \left[\ln(x^2+2)^2\right]^3$

e) $y + \ln(xy) = 1$

f) $y = \dfrac{(2x^3-4)^{\frac{2}{3}} . (2x^2+3)^{\frac{5}{2}}}{(7x^3+4x-3)^2}$

Email: firas.thair.almaliky@uomus.edu.iq

$$a)\ y=\log_{10}e^{x} \Rightarrow y=x\log_{10}e \Rightarrow \frac{dy}{dx}=\log_{10}e=\frac{\ln e}{\ln 10}=\frac{1}{\ln 10}$$

$$b)\ y=\log_{5}(x+1)^{2}=2\log_{5}(x+1) \Rightarrow \frac{dy}{dx}=\frac{2}{(x+1)\ln 5}$$

$$c)\ y=3\log_{2}(3x^{2}+1) \Rightarrow \frac{dy}{dx}=\frac{3}{3x^{2}+1}.\frac{6x}{\ln 2}=\frac{18x}{(3x^{2}+1)\ln 2}$$

$$d)\ \frac{dy}{dx}=3\left[2\ln(x^{2}+2)\right]^{2}\frac{2}{x^{2}+2}.2x=\frac{48x\left[\ln(x^{2}+2)\right]^{2}}{x^{2}+2}$$

$$e)\ y+\ln x+\ln y=1 \Rightarrow \frac{dy}{dx}+\frac{1}{x}+\frac{1}{y}.\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x(y+1)}$$

$$f)\ \ln y=\frac{2}{3}\ln(2x^{3}-4)+\frac{5}{2}\ln(2x^{2}+3)-2\ln(7x^{3}+4x-3)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{2}{3}.\frac{6x^{2}}{2x^{3}-4}+\frac{5}{2}.\frac{4x}{2x^{2}+3}-2.\frac{21x^{2}+4}{7x^{3}+4x-3}$$

$$\Rightarrow \frac{dy}{dx}=2y\left[\frac{2x^{2}}{2x^{3}-4}+\frac{5x}{2x^{2}+3}-\frac{21x^{2}+4}{7x^{3}+4x-3}\right]$$

Inverse Trigonometric Functions

TABLE Derivatives of the inverse trigonometric functions

1. $\dfrac{d(\sin^{-1}u)}{dx}=\dfrac{du/dx}{\sqrt{1-u^{2}}}, \quad |u|<1$

2. $\dfrac{d(\cos^{-1}u)}{dx}=-\dfrac{du/dx}{\sqrt{1-u^{2}}}, \quad |u|<1$

3. $\dfrac{d(\tan^{-1}u)}{dx}=\dfrac{du/dx}{1+u^{2}}$

4. $\dfrac{d(\cot^{-1}u)}{dx}=-\dfrac{du/dx}{1+u^{2}}$

5. $\dfrac{d(\sec^{-1}u)}{dx}=\dfrac{du/dx}{|u|\sqrt{u^{2}-1}}, \quad |u|>1$

6. $\dfrac{d(\csc^{-1}u)}{dx}=\dfrac{-du/dx}{|u|\sqrt{u^{2}-1}}, \quad |u|>1$

Email: firas.thair.almaliky@uomus.edu.iq

EX-1. - Find $\frac{dy}{dx}$ in each of the following functions :

a) $y = \cot^{-1}\frac{2}{x} + \tan^{-1}\frac{x}{2}$ b) $y = \sin^{-1}\frac{x-1}{x+1}$

c) $y = x.\cos^{-1}2x - \frac{1}{2}\sqrt{1-4x^2}$ d) $y = \sec^{-1}5x$

e) $y = x.\ln(\sec^{-1}x)$ f) $y = 3^{\sin^{-1}2x}$

Sol. –

a) $$\frac{dy}{dx} = -\frac{1}{1+\left(\frac{2}{x}\right)^2}2.\left(-\frac{1}{x^2}\right) + \frac{1}{1+\left(\frac{x}{2}\right)^2}.\frac{1}{2} = \frac{4}{4+x^2}$$

b) $$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}}.\frac{(x+1).1-(x-1).1}{(x+1)^2} = \frac{1}{(x+1)\sqrt{x}}$$

c) $$\frac{dy}{dx} = x\frac{-2}{\sqrt{1-4x^2}} + \cos^{-1}2x - \frac{1}{4}.\frac{-8x}{\sqrt{1-4x^2}} = \cos^{-1}2x$$

d) $$\frac{dy}{dx} = \frac{5}{|5x|\sqrt{25x^2-1}} = \frac{1}{|x|\sqrt{25x^2-1}}$$

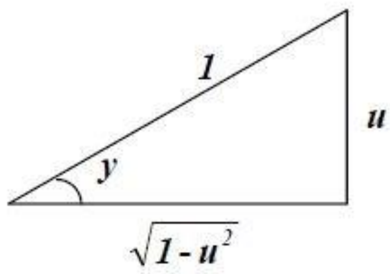
e) $$\frac{dy}{dx} = \frac{x}{\sec^{-1}x}\frac{1}{|x|\sqrt{x^2-1}} + \ln(\sec^{-1}x) = \frac{1}{\sqrt{x^2-1}.\sec^{-1}x} + \ln(\sec^{-1}x)$$

f) $$\frac{dy}{dx} = 3^{\sin^{-1}2x}.\ln 3.\frac{2}{\sqrt{1-4x^2}}$$

EX- 2- Prove that :

a) $\frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$ b) $\frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}\frac{du}{dx}$

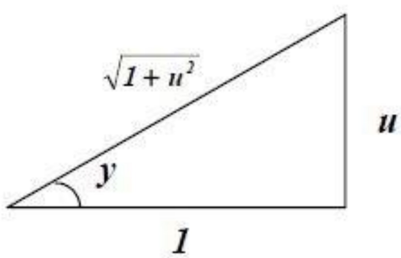
Proof: a)



$$Let\ \ y = \sin^{-1}u \Rightarrow u = \sin y \Rightarrow \frac{du}{dx} = \cos y.\frac{dy}{dx} = \sqrt{1-u^2}\ \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx} \Rightarrow \frac{d}{dx}\sin^{-1}u = \frac{1}{\sqrt{1-u^2}}\frac{du}{dx}$$

b)



$$Let\ y = \tan^{-1}u \Rightarrow u = \tan y \Rightarrow \frac{du}{dx} = \sec^2 y\,.\frac{du}{dx} = \left(\sqrt{1+u^2}\right)^2 \frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{1+u^2}.\frac{du}{dx} \Rightarrow \frac{d}{dx}\tan^{-1}u = \frac{1}{1+u^2}.\frac{du}{dx}$$

Hyperbolic Functions

TABLE Derivatives of hyperbolic functions

$$\frac{d}{dx}(\sinh u) = \cosh u \frac{du}{dx}$$

$$\frac{d}{dx}(\cosh u) = \sinh u \frac{du}{dx}$$

$$\frac{d}{dx}(\tanh u) = \operatorname{sech}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\coth u) = -\operatorname{csch}^2 u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{sech} u) = -\operatorname{sech} u \tanh u \frac{du}{dx}$$

$$\frac{d}{dx}(\operatorname{csch} u) = -\operatorname{csch} u \coth u \frac{du}{dx}$$

TABLE Derivatives of inverse hyperbolic functions

$$\frac{d(\sinh^{-1} u)}{dx} = \frac{1}{\sqrt{1+u^2}}\frac{du}{dx}$$

$$\frac{d(\cosh^{-1} u)}{dx} = \frac{1}{\sqrt{u^2-1}}\frac{du}{dx}, \qquad u > 1$$

$$\frac{d(\tanh^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| < 1$$

$$\frac{d(\coth^{-1} u)}{dx} = \frac{1}{1-u^2}\frac{du}{dx}, \qquad |u| > 1$$

$$\frac{d(\operatorname{sech}^{-1} u)}{dx} = \frac{-du/dx}{u\sqrt{1-u^2}}, \qquad 0 < u < 1$$

$$\frac{d(\operatorname{csch}^{-1} u)}{dx} = \frac{-du/dx}{|u|\sqrt{1+u^2}}, \qquad u \neq 0$$

Email: firas.thair.almaliky@uomus.edu.iq

EX- - Find $\dfrac{dy}{dx}$ for the following functions :

a) $y = \cosh^{-1}(\sec x)$ b) $y = \tanh^{-1}(\cos x)$

c) $y = \coth^{-1}(\sec x)$ d) $y = \operatorname{sech}^{-1}(\sin 2x)$

Sol.-

a) $\dfrac{dy}{dx} = \dfrac{\sec x.\tan x}{\sqrt{\sec^2 x - 1}} = \dfrac{\sec x.\tan x}{\sqrt{\tan^2 x}} = \sec x \quad \text{where } \tan x > 0$

b) $\dfrac{dy}{dx} = \dfrac{-\sin x}{1 - \cos^2 x} = \dfrac{-\sin x}{\sin^2 x} = -\csc x$

c) $\dfrac{dy}{dx} = \dfrac{\sec x.\tan x}{1 - \sec^2 x} = \dfrac{\sec x.\tan x}{-\tan^2 x} = -\csc x$

d) $\dfrac{dy}{dx} = -\dfrac{2.\cos 2x}{\sin 2x.\sqrt{1 - \sin^2 2x}} = -2\csc 2x \quad \text{where } \cos 2x > 0$

EX- – Verify the following formulas :

a) $\dfrac{d}{dx}\cosh^{-1} u = \dfrac{1}{\sqrt{u^2 - 1}} \cdot \dfrac{du}{dx}$

b) $\dfrac{d}{dx}\tanh^{-1} u = \dfrac{1}{1 - u^2} \cdot \dfrac{du}{dx} \quad |u| < 1$

Proof

a) Let $y = \cosh^{-1} u \Rightarrow u = \cosh y$

$$\frac{du}{dx} = \sinh y.\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\sinh y}\cdot\frac{du}{dx}$$

$$\cosh^2 y - \sinh^2 y = 1 \Rightarrow u^2 - \sinh^2 y = 1 \Rightarrow \sinh y = \sqrt{u^2 - 1}$$

$$\frac{dy}{dx} = \frac{1}{\sqrt{u^2 - 1}}\cdot\frac{du}{dx} \Rightarrow \frac{d}{dx}\cosh^{-1} u = \frac{1}{\sqrt{u^2 - 1}}\cdot\frac{du}{dx}$$

b) Let $y = \tanh^{-1} u \Rightarrow u = \tanh y$

$$\frac{du}{dx} = \operatorname{sech}^2 y.\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{1}{\operatorname{sech}^2 y}\cdot\frac{du}{dx}$$

$$\operatorname{sech}^2 y + \tanh^2 y = 1 \Rightarrow \operatorname{sech}^2 y + u^2 = 1 \Rightarrow \operatorname{sech}^2 y = 1 - u^2$$

$$\frac{dy}{dx} = \frac{1}{1 - u^2}\cdot\frac{du}{dx} \Rightarrow \frac{d}{dx}\tanh^{-1} u = \frac{1}{1 - u^2}\cdot\frac{du}{dx}$$

Problems

1. Find $\frac{dy}{dx}$ for the following functions :

1) $y=(x-3)(1-x)$ $\quad (ans.: 4-2x)$

2) $y=\frac{ax+b}{x}$ $\quad (ans.: -\frac{b}{x^2})$

3) $y=\frac{3x+4}{2x+3}$ $\quad (ans.: \frac{1}{(2x+3)^2})$

4) $y=3x^3-2\sqrt{x}+\frac{5}{x^2}$ $\quad (ans.: 9x^2-\frac{1}{\sqrt{x}}-\frac{10}{x^3})$

5) $y=\left(\sqrt{x^3}-\frac{1}{\sqrt{x^3}}\right)^2$ $\quad (ans.: \frac{3(x^6-1)}{x^4})$

6) $y=(2x-1)^2(3x+2)^3+\frac{1}{(x-2)^2}$ $\quad (ans.: (2x-1)(3x+2)^2(30x-1)-\frac{2}{(x-2)^3})$

7) $y=\ln(\ln x)$ $\quad (ans.: \frac{1}{x.\ln x})$

8) $y=\ln(Cosx)$ $\quad (ans.: -\tan x)$

9) $y=Sinx^3$ $\quad (ans.: 3x^2.Cosx^3)$

10) $y=Cos^{-3}(5x^2+2)$ $\quad (ans.: \frac{30x.Sin(5x^2+4)}{Cos^4(5x^2+4)})$

11) $y=\tan x.\sin x$ $\quad (ans.: Sinx+\tan x.Secx)$

12) $y=\tan(Secx)$ $\quad (ans.: Sec^2(Secx).Secx.\tan x)$

13) $y=Cot^3\left(\frac{x+1}{x-1}\right)$ $\quad (ans.: \frac{6}{(x-1)^2}.Cot^2\left(\frac{x+1}{x-1}\right).Csc^2\left(\frac{x+1}{x-1}\right))$

14) $y=\frac{Cosx}{x}$ $\quad (ans.: -\frac{x.Sinx+Cosx}{x^2})$

15) $y=\sqrt{\tan\sqrt{2x+7}}$ $\quad (ans.: \frac{Sec^2\sqrt{2x+7}}{2\sqrt{2x+7}\sqrt{\tan\sqrt{2x+7}}})$

16) $y=x^2.Sinx$ $\quad (ans.: x^2.Cosx+2x.Sinx)$

17) $y=Csc^{-\frac{2}{3}}\sqrt{5x}$ $\quad (ans.: \frac{5}{3\sqrt{5x}}.\frac{Cot\sqrt{5x}}{Csc^{\frac{2}{3}}\sqrt{5x}})$

18) $y=x[Sin(\ln x)+Cos(\ln x)]$ $\quad (ans.: 2.Cos(\ln x))$

Email: firas.thair.almaliky@uomus.edu.iq

19) $y = Sin^{-1}(5x^2)$ $\quad (ans.: \dfrac{10x}{\sqrt{1-25x^4}})$

20) $y = Cot^{-1}\left(\dfrac{1+x}{1-x}\right)$ $\quad (ans.: -\dfrac{1}{1+x^2})$

21) $y = tan^{-1}\sqrt{4x^3-2}$ $\quad (ans.: \dfrac{6x^2}{(4x^3-1)\sqrt{4x^3-2}})$

22) $y = Sec^{-1}(3x^2+1)^3$ $\quad (ans.: \dfrac{18x}{|3x^2+1|\sqrt{(3x^2+1)^6-1}})$

23) $y = Sin^{-1}\dfrac{x^2}{2-x} + x^2.Sec^{-1}\dfrac{x}{2}$ $\quad (ans.: \dfrac{4x-x^2}{(2-x)\sqrt{(2-x)^2-x^4}} + \dfrac{2x}{\sqrt{x^2-4}} + 2x.Sec^{-1}\dfrac{x}{2})$

24) $y = Sin^{-1}2x.Cos^{-1}2x$ $\quad (ans.: \dfrac{2(Cos^{-1}2x - Sin^{-1}2x)}{\sqrt{1-4x^2}})$

25) $y = \sqrt[3]{\dfrac{x(x+1)(x-2)}{(x^2+1)(2x+3)}}$ $\quad (ans.: \dfrac{y}{3}\left[\dfrac{1}{x} + \dfrac{1}{x+1} + \dfrac{1}{x-2} - \dfrac{2x}{x^2+1} - \dfrac{2}{2x+3}\right])$

26) $y = tan^{-1}(\ln x)$ $\quad (ans.: \dfrac{1}{x(1+(\ln x)^2)})$

27) $y^{\frac{4}{3}} = \dfrac{\sqrt{\sin x.\cos x}}{1+2.\ln x}$ $\quad (ans.: \dfrac{3y}{4}(\dfrac{\cot x}{2} - \dfrac{\tan x}{2} - \dfrac{2}{x(1+2\ln x)}))$

28) $\sqrt{y} = \dfrac{x^5.\tan^{-1}x}{(3-2x).\sqrt[3]{x}}$ $\quad (ans.: 2y(\dfrac{14}{3x} + \dfrac{1}{(1+x^2).\tan^{-1}x} + \dfrac{2}{3-2x}))$

29) $y = sec^{-1}e^{2x}$ $\quad (ans.: \dfrac{2}{\sqrt{e^{4x}-1}})$

30) $y = (\cos x)^{\sqrt{x}}$ $\quad (ans.: \dfrac{y}{2\sqrt{x}}(\ln\cos x - 2x.\tan x))$

31) $y = (\sin x)^{\tan x}$ $\quad (ans.: y(1+sec^2 x.\ln\sin x))$

32) $y = \sqrt{2x^2 + cosh^2(5x)}$ $\quad (ans.: \dfrac{2x+5\cosh(5x).\sinh(5x)}{\sqrt{2x^2+cosh^2(5x)}})$

33) $y = \sinh(\cos 2x)$ $\quad (ans.: -2\sin 2x.\cosh(\cos 2x))$

34) $y = csch\dfrac{1}{x}$ $\quad (ans.: \dfrac{1}{x^2}.csch\dfrac{1}{x}.coth\dfrac{1}{x})$

35) $y = x^2.tanh^2\sqrt{x}$ $\quad (ans.: x.tanh\sqrt{x}(\sqrt{x}\,sech^2\sqrt{x} + 2tanh\sqrt{x}))$

2. Verify the following derivatives :

a) $\dfrac{d}{dx}\left[5x+(\sqrt{x}+\dfrac{1}{\sqrt{x}})^2\right]=6-\dfrac{1}{x^2}$

b) $\dfrac{d}{dx}\left[\sqrt{x}(ax^2+bx+c)\right]=\dfrac{1}{2\sqrt{x}}(5ax^2+3bx+c)$

3. Find the derivative of y with respect to x in the following functions :

a) $y=\dfrac{u^2}{u^2+1}$ and $u=3x^3-2$ $\quad$ (ans.: $\dfrac{18x^2y^2}{(3x^3-2)^3}$)

b) $y=\sqrt{u}+2u$ and $u=x^2-3$ $\quad$ (ans.: $\dfrac{x}{\sqrt{x^2-3}}+4x$)

4. Find the second derivative for the following functions :

a) $y=(x+\dfrac{1}{x})^3$ $\quad$ (ans.: $6x+\dfrac{6}{x^3}+\dfrac{12}{x^5}$)

b) $f(x)=\sqrt{2x}+\dfrac{2\sqrt{2}}{\sqrt{x}}$ at $x=2$ $\quad$ (ans.: $\dfrac{1}{4}$)

Find $\dfrac{dy}{dx}$ for the following implicit functions :

a) $x^3+4x\sqrt{y}-\dfrac{5y^2}{x}=3$ $\quad$ (ans.: $\dfrac{3x^2+5y^2x^{-2}+4\sqrt{y}}{10x^{-1}y-\dfrac{2x}{\sqrt{y}}}$)

b) $\sqrt{xy}+1=y$ $\quad$ (ans.: $\dfrac{y}{2\sqrt{xy}-x}$)

c) $3xy=(x^3+y^3)^{\frac{3}{2}}$ $\quad$ (ans.: $\dfrac{3x^2\sqrt{x^3+y^3}-2y}{2x-3y^2\sqrt{x^3+y^3}}$)

d) $x^3+x.\tan^{-1}y=y$ $\quad$ (ans.: $\dfrac{(1+y^2)(3x^2+\tan^{-1}y)}{1+y^2-x}$)

e) $\sin^{-1}(xy)=\cos^{-1}(x-y)$ $\quad$ (ans.: $\dfrac{y\sqrt{1-(x-y)^2}+\sqrt{1-(xy)^2}}{\sqrt{1-(xy)^2}-x\sqrt{1-(x-y)^2}}$)

f) $y^2.\sin(xy)=\tan x$ $\quad$ (ans.: $\dfrac{\sec^2 x-y^3.\cos(xy)}{2y.\sin(xy)+xy^2.\cos(xy)}$)

g) $\sinh y=\tan^2 x$ $\quad$ (ans.: $\dfrac{2.\tan x.\sec^2 x}{\cosh y}$)

Email: firas.thair.almaliky@uomus.edu.iq