



Chain Rule :

Lecture five

The chain rule is used to differentiate the composite functions.

1. Chain rule for function of single variable defined along paths.

$$y = f(x)$$

$$x = x(t) \Rightarrow \frac{dy}{dt} = \frac{dy}{dx} * \frac{dx}{dt}$$

$$y = F[x(t)]$$

Ex: if $y = x^2 + 1$, $x = \tan^{-1} t$, find $\frac{dy}{dt}$?

$$\text{Sol: } \frac{dy}{dx} = 2x$$

$$\frac{dx}{dt} = \frac{1}{1+t^2}$$

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

$$= 2x \cdot \frac{1}{1+t^2} = 2 \tan^{-1} t * \frac{1}{1+t^2}$$



2. For the function $Z = f(x, y)$ of two variable defined along path.

$$Z = f[x(t), y(t)]$$

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}$$

Ex: $Z = f(x, y) = x^2 y^3$, $x = \cos t$, $y = \sin t$ find $\frac{df}{dt}$?

Sol:

$$\frac{df}{dt} = \frac{df}{dx} \cdot \frac{dx}{dt} + \frac{df}{dy} \cdot \frac{dy}{dt}$$

$$f_x = y^3 \cdot 2x = 2xy^3$$

$$f_y = 3y^2 x^2$$

$$\frac{dy}{dt} = \cos t, \quad \frac{dx}{dt} = -\sin t$$

$$\frac{df}{dt} = 2xy^3(-\sin t) + 3y^2x^2(\cos t)$$

$$= 2 \cos t (\sin t)^3 (-\sin t) + 3 (\sin t)^2 (\cos t)^2 (\cos t)$$

$$= -2 \cos t (\sin t)^4 + 3 (\sin t)^2 (\cos t)^3$$



3. Chain rule for function to more than two variable defined along path :

$$w = f(x_1, x_2, x_3, \dots, x_n)$$

$$\frac{df}{dt} = \frac{df}{dx_1} \cdot \frac{dx_1}{dt} + \frac{df}{dx_2} \cdot \frac{dx_2}{dt} + \dots + \frac{df}{dx_n} \cdot \frac{dx_n}{dt}$$

Ex: let $w = xy \sin z$ where $x = \cos t$, $y = \sin t$, $z = 1 + t^2$

find $\frac{dw}{dt}$?

Sol:

$$\frac{df}{dt} = \frac{df}{dx_1} \cdot \frac{dx_1}{dt} + \frac{df}{dx_2} \cdot \frac{dx_2}{dt} + \frac{df}{dx_3} \cdot \frac{dx_3}{dt}$$

$$F_x = y \sin z, F_y = x \sin z, F_z = xy \cos z$$

$$\frac{dx}{dt} = -\sin t$$

$$\frac{dy}{dt} = \cos t$$

$$\frac{dz}{dt} = 2t$$

$$\frac{dw}{dt} = y \sin z (-\sin t) + x \sin z (\cos t) + xy \cos z (2t)$$

$$= y \sin(1+t^2) (-\sin t) + \cos t \sin(1+t^2) x + \cos t \sin t \cos(1+t^2) (2t)$$



4. Chain rule for function of two variable defined on surface :

If $z = f(x, y)$ has partial derivatives $\frac{\partial f}{\partial x}$, $\frac{\partial f}{\partial y}$ and if

$$x = x(r, s) \text{ and } y = y(r, s)$$

$$z = f[x(r, s), y(r, s)]$$

$$\frac{\partial z}{\partial r} = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s}$$

Ex: find $\frac{\partial z}{\partial r}$ and $\frac{\partial z}{\partial s}$ if $z = f(x, y) = x^2 + y^2$

$$x = r + e^s, \quad y = \ln s ?$$

Sol:

$$\frac{\partial z}{\partial r} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial r}$$

$$f_x = 2x, \quad \frac{\partial x}{\partial r} = 1, \quad \frac{\partial y}{\partial r} = 0$$

$$f_y = 2y, \quad \frac{\partial x}{\partial s} = e^s, \quad \frac{\partial y}{\partial s} = \frac{1}{s}$$

$$\frac{\partial z}{\partial r} = 2x(1) + 2y(0) = 2x = 2(r + e^s)$$

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s}$$



$$\begin{aligned} &= 2x e^s + 2y (1/s) \\ &= 2(r+e^s) \cdot e^s + 2(\ln s) \cdot \frac{1}{s} \\ &= 2r e^s + 2e^{2s} + \frac{2}{s} \ln s \end{aligned}$$

Ex: If z is a differentiable function of x and y which satisfy the equation $x^3 + y^3 + z^3 + 3x^2 \sin y \tan z = 5$?
Find $\frac{\partial z}{\partial x}$, $\frac{\partial z}{\partial y}$?

Sol:

$$3x^2 + 0 + 3z^2 \frac{\partial z}{\partial x} + 3 \sin y (x^2 \sec^2 z \frac{\partial z}{\partial x} + \tan z \cdot 2x) = 0$$

$$3x^2 + 3z^2 \frac{\partial z}{\partial x} + 3x^2 \sin y \sec^2 z \frac{\partial z}{\partial x} + 6x \sin y \tan z = 0$$

$$3x^2 \sin y \sec^2 z \frac{\partial z}{\partial x} + 3z^2 \frac{\partial z}{\partial x} = -3x^2 - 6x \sin y \tan z$$

$$\frac{\partial z}{\partial x} (3x^2 \sin y \sec^2 z + 3z^2) = -3x^2 - 6x \sin y \tan z$$

$$\frac{\partial z}{\partial x} = \frac{-3x^2 - 6x \sin y \tan z}{3x^2 \sin y \sec^2 z + 3z^2}$$



$$0 + 3y^2 + 3z^2 \frac{\partial z}{\partial y} + 3x^2 (\sin y \sec^2 z \frac{\partial z}{\partial y} + \tan z \cos y) = 0$$

$$3y^2 + 3z^2 \frac{\partial z}{\partial y} + 3x^2 \sin y \sec^2 z \frac{\partial z}{\partial y} + 3x^2 \tan z \cos y = 0$$

$$3z^2 \frac{\partial z}{\partial y} + 3x^2 \sin y \sec^2 z \frac{\partial z}{\partial y} = -3y^2 - 3x^2 \tan z \cos y$$

$$\frac{\partial z}{\partial y} (3z^2 + 3x^2 \sin y \sec^2 z) = -3y^2 - 3x^2 \tan z \cos y$$

$$\frac{\partial z}{\partial y} = \frac{-3y^2 - 3x^2 \tan z \cos y}{3z^2 + 3x^2 \sin y \sec^2 z}$$

5. Chain rule for function of three variables defined on surface :-

If $w = f(x, y, z)$ has partial derivatives

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ and if $x = x(r, s), z = z(r, s)$

also have partial derivatives

$$w = f[x(r, s), y(r, s), z(r, s)]$$

$$\frac{\partial w}{\partial r} = \frac{\partial f}{\partial r} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$\frac{\partial w}{\partial s} = \frac{\partial f}{\partial s} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial s} + \frac{\partial f}{\partial z} \cdot \frac{\partial z}{\partial s}$$



Ex: Find $\frac{dw}{dr}$ if $w = f(x, y, z) = x + 2y + z^2$ where $x = \frac{r}{5}$
 $y = r^2 + e^s$, $z = 2r$?

Sol:

$$w = x + 2y + z^2$$

$$\frac{\partial w}{\partial x} = 1 + 0 + 0 = 1$$

$$\frac{\partial w}{\partial y} = 0 + 2 + 0 = 2$$

$$\frac{\partial w}{\partial z} = 0 + 0 + 2z = 2z$$

$$x = \frac{r}{5} \Rightarrow \frac{\partial x}{\partial r} = \frac{1}{5} * 1 = \frac{1}{5}$$

$$y = r^2 + e^s \Rightarrow \frac{\partial y}{\partial r} = 2r + 0 = 2r$$

$$z = 2r \Rightarrow \frac{\partial z}{\partial r} = 2$$

$$\frac{dw}{dr} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial r} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial r} + \frac{\partial w}{\partial z} \cdot \frac{\partial z}{\partial r}$$

$$= 1 * \frac{1}{5} + 2 * 2r + 2z * 2$$

$$= \frac{1}{5} + 4r + 4z$$

$$= \frac{1}{5} + 4r + 4(2r) = \frac{1}{5} + 12r$$