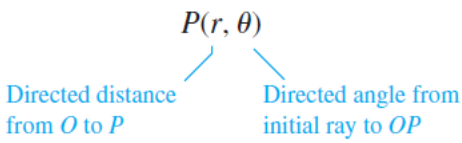
Polar Coordinates

To define polar coordinates, we first fix an **origin *O*** (called the **pole**) and an **initial ray** from ***O*** (Figure 1). Usually the positive *x*-axis is chosen as the initial ray. Then each point *P* can be located by assigning to it a **polar coordinate pair** (r, θ) in which r gives the directed distance from O to P and θ gives the directed angle from the initial ray to ray OP. So we label the point P as;



For instance, the point 2 units from the origin along the ray $\theta = \pi/6$ has polar coordinates $r = 2$, $\theta = \pi/6$. It also has coordinates $r = 2$, $\theta = -11\pi/6$ (Figure 2).

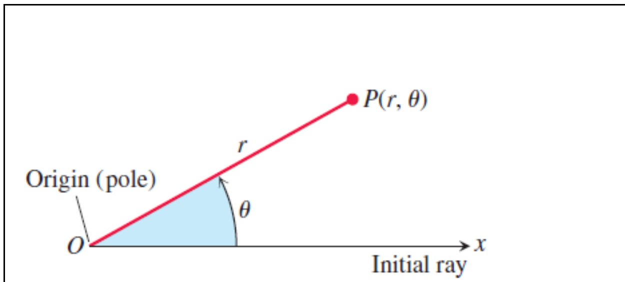


Figure 1

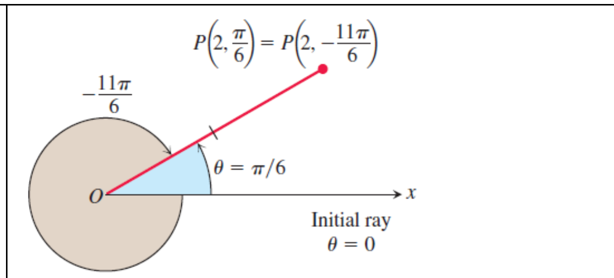


Figure 2

The usual way to relate polar and Cartesian coordinates is shown in Figure 3 and is determined using the equation below:

Equations Relating Polar and Cartesian Coordinates

$$x = r\cos\theta, \qquad y = r\sin\theta, \qquad r^2 = x^2 + y^2, \qquad \tan\theta = \frac{y}{x}$$

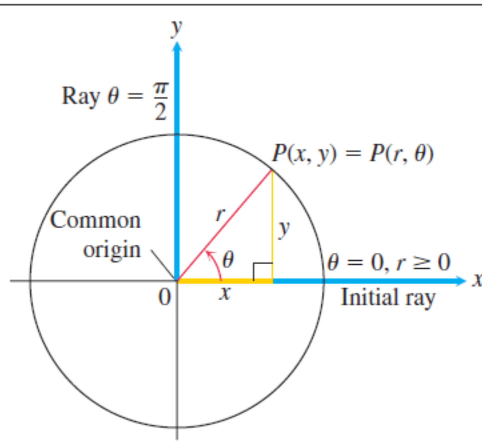


Figure 3

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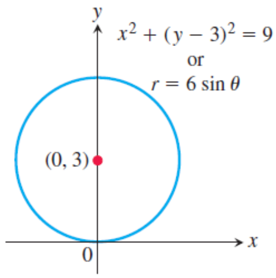

EXAMPLE Find a polar equation for the circle $x^2 + (y-3)^2 = 9$

$$x^2 + (y-3)^2 = 9$$
$$x^2 + y^2 - 6y + 9 = 9 \quad \text{Expand } (y-3)^2.$$
$$x^2 + y^2 - 6y = 0 \quad \text{Cancelation}$$
$$r^2 - 6r\sin\theta = 0 \quad x^2 + y^2 = r^2,\ y = r\sin\theta$$
$$r = 0 \quad \text{or} \quad r - 6\sin\theta = 0$$
$$r = 6\sin\theta \quad \text{Includes both possibilities}$$

EXAMPLE: Replace the following polar equations by equivalent Cartesian equations and identify their graphs.

(a) $r\cos\theta = -4$
(b) $r^2 = 4r\cos\theta$
(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

Solution We use the substitutions $r\cos\theta = x$, $r\sin\theta = y$, and $r^2 = x^2 + y^2$.

(a) $r\cos\theta = -4$

The Cartesian equation: $r\cos\theta = -4$
$$x = -4 \quad \text{Substitution}$$
The graph: Vertical line through $x = -4$ on the x-axis

(b) $r^2 = 4r\cos\theta$

The Cartesian equation: $r^2 = 4r\cos\theta$
$$x^2 + y^2 = 4x \quad \text{Substitution}$$
$$x^2 - 4x + y^2 = 0$$
$$x^2 - 4x + 4 + y^2 = 4 \quad \text{Completing the square}$$
$$(x-2)^2 + y^2 = 4 \quad \text{Factoring}$$
The graph: Circle, radius 2, center $(h, k) = (2, 0)$

(c) $r = \dfrac{4}{2\cos\theta - \sin\theta}$

The Cartesian equation: $r(2\cos\theta - \sin\theta) = 4$
$$2r\cos\theta - r\sin\theta = 4 \quad \text{Multiplying by } r$$
$$2x - y = 4 \quad \text{Substitution}$$
$$y = 2x - 4 \quad \text{Solve for } y.$$

EXAMPLE Here are some plane curves expressed in terms of both polar coordinate and Cartesian coordinate equations.

Polar equation	Cartesian equivalent
$r\cos\theta = 2$	$x = 2$
$r^2\cos\theta\sin\theta = 4$	$xy = 4$
$r^2\cos^2\theta - r^2\sin^2\theta = 1$	$x^2 - y^2 = 1$
$r = 1 + 2r\cos\theta$	$y^2 - 3x^2 - 4x - 1 = 0$
$r = 1 - \cos\theta$	$x^4 + y^4 + 2x^2y^2 + 2x^3 + 2xy^2 - y^2 = 0$

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