Derivatives of Trigonometric Functions

The derivative of the sine function is the cosine function:

$$\frac{d}{dx}(\sin x) = \cos x.$$

The derivative of the cosine function is the negative of the sine function:

$$\frac{d}{dx}(\cos x) = -\sin x.$$

The derivatives of the other trigonometric functions:

$$\frac{d}{dx}(\tan x) = \sec^2 x \qquad \frac{d}{dx}(\cot x) = -\csc^2 x$$

$$\frac{d}{dx}(\sec x) = \sec x \tan x \qquad \frac{d}{dx}(\csc x) = -\csc x \cot x$$

Many phenomena of nature are approximately periodic (electromagnetic fields, heart rhythms, tides, weather). The derivatives of sines and cosines play a key role in describing periodic changes. This section shows how to differentiate the six basic trigonometric functions.

EXAMPLE 1 Derivative of the Sine Function

To calculate the derivative of $f(x) = \sin x$, for x measured in radians, we combine the limits with the angle sum identity for the sine function:

If $f(x) = \sin x$, then $\sin(x + h) = \sin x \cos h + \cos x \sin h$.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sin(x+h) - \sin x}{h} \qquad \text{Derivative definition}$$

$$= \lim_{h \to 0} \frac{(\sin x \cos h + \cos x \sin h) - \sin x}{h} = \lim_{h \to 0} \frac{\sin x(\cos h - 1) + \cos x \sin h}{h}$$

$$= \lim_{h \to 0} \left(\sin x \cdot \frac{\cos h - 1}{h}\right) + \lim_{h \to 0} \left(\cos x \cdot \frac{\sin h}{h}\right)$$

$$= \sin x \cdot \underbrace{\lim_{h \to 0} \frac{\cos h - 1}{h}}_{\text{limit 0}} + \cos x \cdot \underbrace{\lim_{h \to 0} \frac{\sin h}{h}}_{\text{limit 1}} = \sin x \cdot 0 + \cos x \cdot 1 = \cos x.$$

EXAMPLE 2 Derivative of the Cosine Function

With the help of the angle sum formula for the cosine function,

$$\cos(x+h) = \cos x \cos h - \sin x \sin h,$$

we can compute the limit of the difference quotient:

$$\begin{aligned}
\frac{d}{dx}(\cos x) &= \lim_{h\to 0} \frac{\cos(x+h) - \cos x}{h} && \text{Derivative definition}\\
&= \lim_{h\to 0} \frac{(\cos x \cos h - \sin x \sin h) - \cos x}{h} && \text{Cosine angle sum identity}\\
&= \lim_{h\to 0} \frac{\cos x(\cos h - 1) - \sin x \sin h}{h}\\
&= \lim_{h\to 0} \cos x \cdot \frac{\cos h - 1}{h} - \lim_{h\to 0} \sin x \cdot \frac{\sin h}{h}\\
&= \cos x \cdot \lim_{h\to 0} \frac{\cos h - 1}{h} - \sin x \cdot \lim_{h\to 0} \frac{\sin h}{h}\\
&= \cos x \cdot 0 - \sin x \cdot 1\\
&= -\sin x.
\end{aligned}$$

EXAMPLE 3 We find derivatives of the function involving differences, products, and quotients.

(a) $y = x^2 - \sin x$:

$$\begin{aligned}
\frac{dy}{dx} &= 2x - \frac{d}{dx}(\sin x) && \text{Difference Rule}\\
&= 2x - \cos x
\end{aligned}$$

(b) $y = \frac{\sin x}{x}$:

$$\begin{aligned}
\frac{dy}{dx} &= \frac{x \cdot \frac{d}{dx}(\sin x) - \sin x \cdot 1}{x^2} && \text{Quotient Rule}\\
&= \frac{x \cos x - \sin x}{x^2}
\end{aligned}$$

(c) $y = \sin x \cos x$:

$$\begin{aligned}
\frac{dy}{dx} &= \sin x \frac{d}{dx}(\cos x) + \cos x \frac{d}{dx}(\sin x) && \text{Product Rule}\\
&= \sin x(-\sin x) + \cos x(\cos x)\\
&= \cos^2 x - \sin^2 x
\end{aligned}$$

(e) $y = \dfrac{\cos x}{1 - \sin x}$:

$$\frac{dy}{dx} = \frac{(1 - \sin x)\frac{d}{dx}(\cos x) - \cos x \frac{d}{dx}(1 - \sin x)}{(1 - \sin x)^2} \qquad \text{Quotient Rule}$$

$$= \frac{(1 - \sin x)(-\sin x) - \cos x(0 - \cos x)}{(1 - \sin x)^2}$$

$$= \frac{1 - \sin x}{(1 - \sin x)^2} \qquad \sin^2 x + \cos^2 x = 1$$

$$= \frac{1}{1 - \sin x}$$

EXAMPLE 4 Find $d(\tan x)/dx$.

Solution We use the Derivative Quotient Rule to calculate the derivative:

$$\frac{d}{dx}(\tan x) = \frac{d}{dx}\left(\frac{\sin x}{\cos x}\right) = \frac{\cos x \frac{d}{dx}(\sin x) - \sin x \frac{d}{dx}(\cos x)}{\cos^2 x} \qquad \text{Quotient Rule}$$

$$= \frac{\cos x \cos x - \sin x(-\sin x)}{\cos^2 x}$$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x}$$

$$= \frac{1}{\cos^2 x} = \sec^2 x.$$

EXAMPLE 5 Find y'' if $y = \sec x$.

Solution Finding the second derivative involves a combination of trigonometric derivatives.

$$y = \sec x$$

$$y' = \sec x \tan x \qquad \text{Derivative rule for secant function}$$

$$y'' = \frac{d}{dx}(\sec x \tan x)$$

$$= \sec x \frac{d}{dx}(\tan x) + \tan x \frac{d}{dx}(\sec x) \qquad \text{Derivative Product Rule}$$

$$= \sec x(\sec^2 x) + \tan x(\sec x \tan x) \qquad \text{Derivative rules}$$

$$= \sec^3 x + \sec x \tan^2 x$$

Exercises 3.5

Derivatives

In Exercises 1–18, find dy/dx.

1. $y = -10x + 3\cos x$
2. $y = \frac{3}{x} + 5\sin x$
3. $y = x^2 \cos x$
4. $y = \sqrt{x}\sec x + 3$
5. $y = \csc x - 4\sqrt{x} + 7$
6. $y = x^2 \cot x - \frac{1}{x^2}$
7. $f(x) = \sin x \tan x$
8. $g(x) = \csc x \cot x$
9. $y = (\sec x + \tan x)(\sec x - \tan x)$
10. $y = (\sin x + \cos x)\sec x$
11. $y = \frac{\cot x}{1 + \cot x}$
12. $y = \frac{\cos x}{1 + \sin x}$
13. $y = \frac{4}{\cos x} + \frac{1}{\tan x}$
14. $y = \frac{\cos x}{x} + \frac{x}{\cos x}$
15. $y = x^2 \sin x + 2x\cos x - 2\sin x$
16. $y = x^2 \cos x - 2x\sin x - 2\cos x$
17. $f(x) = x^3 \sin x \cos x$
18. $g(x) = (2 - x)\tan^2 x$

In Exercises 19–22, find ds/dt.

19. $s = \tan t$
20. $s = t^2 - \sec t$
21. $s = \frac{1 + \csc t}{1 - \csc t}$
22. $s = \frac{\sin t}{1 - \cos t}$

In Exercises 23–26, find $dr/d\theta$.

23. $r = 4 - \theta^2 \sin\theta$
24. $r = \theta\sin\theta + \cos\theta$
25. $r = \sec\theta\csc\theta$
26. $r = (1 + \sec\theta)\sin\theta$

In Exercises 27–32, find dp/dq.

27. $p = 5 + \frac{1}{\cot q}$
28. $p = (1 + \csc q)\cos q$
29. $p = \frac{\sin q + \cos q}{\cos q}$
30. $p = \frac{\tan q}{1 + \tan q}$
31. $p = \frac{q\sin q}{q^2 - 1}$
32. $p = \frac{3q + \tan q}{q\sec q}$
33. Find y'' if
    a. $y = \csc x$.
    b. $y = \sec x$.
34. Find $y^{(4)} = d^4y/dx^4$ if
    a. $y = -2\sin x$.
    b. $y = 9\cos x$.

Tangent Lines

In Exercises 35–38, graph the curves over the given intervals, together with their tangents at the given values of x. Label each curve and tangent with its equation.

35. $y = \sin x, \quad -3\pi/2 \le x \le 2\pi$
    $x = -\pi, 0, 3\pi/2$
36. $y = \tan x, \quad -\pi/2 < x < \pi/2$
    $x = -\pi/3, 0, \pi/3$
37. $y = \sec x, \quad -\pi/2 < x < \pi/2$
    $x = -\pi/3, \pi/4$
38. $y = 1 + \cos x, \quad -3\pi/2 \le x \le 2\pi$
    $x = -\pi/3, 3\pi/2$