(Probability)

Probability : The likelihood that something ***will*** happen.

How can data obtained?

Data are obtained by observing either uncontrolled events in **nature** or by observing events in **controlled** situations. We use the term ***experiment*** to describe either method of data collection.

Some Important Terms :

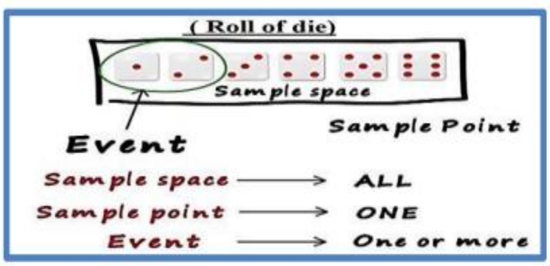
Experiment : is the process by which an observation (or measurement) is obtained.

Outcome : A possible result of one trial of a probability experiment.

Sample Point : is the **one of each** outcome.

Event : is the outcome that is observed on a single repetition of the experiment.

Sample space: is a collection of events. Or, the set of all events.



If an experiment has equally likely outcomes and of these the event A is defined, then the **theoretical probability of event A** occuring is given by

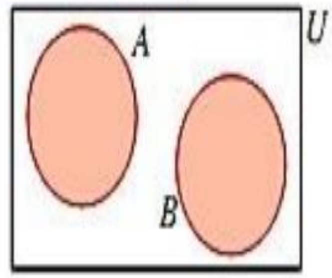
$$P(A) = \frac{n(A)}{n(U)} = \frac{\text{Number of outcomes in which A occurs}}{\text{Total number of outcomes in the sample space}}$$

Where $n(U)$ is the total number of possible outcomes in the sample space, U, (i.e., $n(U) = N$).
As a consequence of this definition we have what are known as the **axioms of probability**:

1. $0 \le P(A) \le 1$
2. $P(\varnothing) = 0$ and $P(\varepsilon) = 1$
   That is, if $A = \varnothing$, then the event A can never occur.
   $A = U$ implies that the event A is a certainty.
3. If A and B are both subsets of U and are mutually exclusive, then
   $P(A \cup B) = P(A) + P(B)$.

Note:

Two events A and B are said to be **mutually exclusive** (or disjoint) if they have no elements in common, i.e., if $A \cap B = \emptyset$.



EXAMPLE

A fair die is thrown. List the sample space of the experiment and hence find the probability of observing: (a) a multiple of 3
(b) an odd number.

Are these events mutually exclusive?

Solution

(a) The sample space is $U = \{1, 2, 3, 4, 5, 6\}$.

Let A be the event 'obtaining a multiple of 3'.

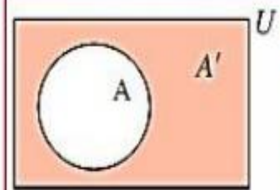
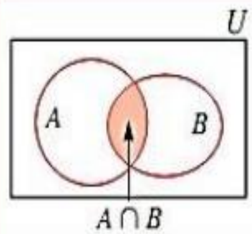
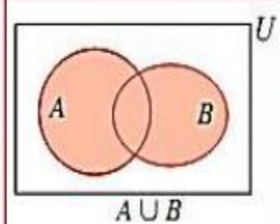
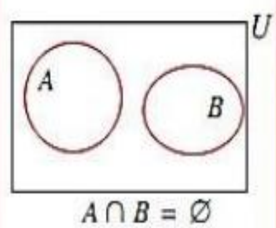
We then have that $A = \{3, 6\}$. Therefore, $P(A) = \frac{n(A)}{n(U)} = \frac{2}{6} = \frac{1}{3}$.

(b) Let B be the event 'obtaining an odd number'.

Here $B = \{1, 3, 5\}$ and so $P(B) = \frac{n(B)}{n(U)} = \frac{3}{6} = \frac{1}{2}$.

In this case, $A = \{3, 6\}$ and $B = \{1, 3, 5\}$, so that $A \cap B = \{3\}$. Therefore, as $A \cap B \neq \emptyset$ A and B are not mutually exclusive.

Engineering Analysis/ Y3 *Probability*

Event	Set language	Venn diagram	Probability result
The **complement** of A is denoted by A'.	A' is the complement to the set A, i.e., the set of elements that do not belong to the set A.		$P(A') = 1 - P(A)$ $P(A')$ is the probability that event A does not occur.
The **intersection** of A and B: $A \cap B$	$A \cap B$ is the intersection of the sets A and B, i.e., the set of elements that belong to **both** the set A **and** the set B.		$P(A \cap B)$ is the probability that both A and B occur.
The **union** of events A and B: $A \cup B$	$A \cup B$ is the union of the sets A and B, i.e., the set of elements that belong to A **or** B or both A **and** B.		$P(A \cup B)$ is the probability that either event A or event B (or both) occur. From this we have what is known as the '**Addition rule**' for probability: $P(A \cup B) = P(A) + P(B) - P(A \cap B)$
If $A \cap B = \varnothing$ the events A and B are said to be **disjoint**. That is, they have no elements in common.	If $A \cap B = \varnothing$ the sets A and B are **mutually exclusive**.		If A and B are mutually exclusive events then event A and event B cannot occur simultaneously, i.e., $n(A \cap B) = 0$ $\Rightarrow P(A \cap B) = 0$ Therefore: $P(A \cup B) = P(A) + P(B)$

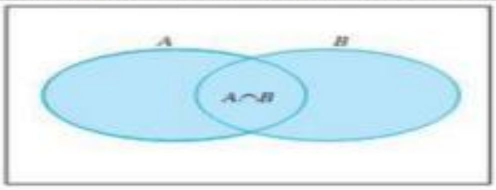
Calculating Probabilities for Unions

When we can write the event of interest in the form of a union, a complement, or an intersection, there are special probability rules that can simplify our calculations. The first rule deals with ***unions of events***.

General addition rule

Given two events, A and B, the probability of their union, $A \cup B$, is equal to

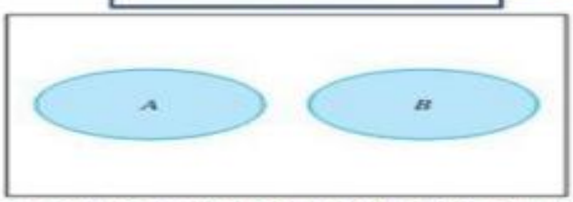
$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$



Special case of additon rule (mutually exclusive)

When two events A and B are **mutually exclusive** or **disjoint,** it means that when A occurs, B cannot, and vice versa. This means that the probability that they both occur, $P(A \cap B)$, must be zero. Figure is a Venn diagram representation of two such events with no simple events in common.

$$P(A \cup B) = P(A) + P(B)$$



When two events A and B are **mutually exclusive,** then $P(A \cap B) = 0$ and the Addition Rule simplifies to

Example :

1	2	3	4	5	6	7	8	9	10

A ⟶ Even B ⟶ Greater than 5

P (A ∪ B) = P (A happening) + P (B happening) − P (A & B happening together)

$= \frac{5}{10} + \frac{5}{10} - \frac{3}{10}$ A --> 2 , 4 , 6 , 8 , 10

$= 0.7$ B --> 6 , 7 , 8 , 9 , 10

2 , 4 , 6 , 7 , 8 , 9 , 10

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

1	2	3	4	5	6

A ⟶ Odd

B ⟶ Even

$P(A \cup B) = P(A) + P(B) - P(A \cap B)$

A & B ---> Mutually Exclusive

$P(A \cup B) = P(A) + P(B)$

$= \frac{3}{6} + \frac{3}{6} = 1$

$P(A \cup B) = P(A) + P(B)$ Mutually Exclusive

EXAMPLE

A bag has 20 coins numbered from 1 to 20. A coin is drawn at random and its number is noted. What is the probability that the coin has a number that is divisible by 3 or by 5?

Solution

Let T denote the event "The number is divisible by 3" and S, the event "The number is divisible by 5".

Using the addition rule we have $P(T \cup S) = P(T) + P(S) - P(T \cap S)$

Now, $T = \{3, 6, 9, 12, 15, 18\}$ and $S = \{5, 10, 15, 20\}$ so that $T \cap S = \{15\}$.

Therefore, we have $P(T) = \frac{6}{20}$ and $P(S) = \frac{4}{20}$ and $P(T \cap S) = \frac{1}{20}$.

This means that $P(T \cup S) = \frac{6}{20} + \frac{4}{20} - \frac{1}{20} = \frac{9}{20}$.

EXAMPLE

If $p(A) = 0.6$, $p(B) = 0.3$ and $p(A \cap B) = 0.2$, find

(a) $p(A \cup B)$ (b) $p(B')$

Solution

(a) Using the addition formula we have, $p(A \cup B) = p(A) + p(B) - p(A \cap B)$

$$\Rightarrow p(A \cup B) = 0.6 + 0.3 - 0.2 = 0.7$$

(b) Using the complementary formula, we have $p(B') = 1 - p(B) = 1 - 0.3 = 0.7$.

Multiplication Rule

Conditional Probability	$P(A \cap B) = P(A \mid B) \times P(B)$
Independence	$P(A \cap B) = P(A) \times P(B)$

⟹ $P(\text{first choice and second choice}) = P(\text{second} \mid \text{first}) \times P(\text{first choice})$

Conditional probability

If the events are *not independent*, one event affects the probability for the other event. In this case *conditional probability* must be used. The conditional probability of B given that A occurs, or *on condition* that A occurs, is written **Pr [B | A]**. This is read as the probability of B given A, or the probability of B on condition that A occurs.

Independence : Two events, A and B, are said to be independent if and only if the probability of event B is not influenced or changed by the occurrence of event A, or vice versa

Example ': A bag contains green balls and yellow balls. You are going to choose two balls without replacement. If the probability of selecting a green ball and a yellow ball is $\frac{14}{39}$, what is the probability of selecting a yellow ball on the second draw, if you know that the probability of selecting a green ball on the first draw is $\frac{4}{9}$.

Solution:

Step 1: List what you know

$$P(Green) = \frac{4}{9}$$

$$P(Green\ AND\ Yellow) = \frac{14}{39}$$

$P(\text{first choice and second choice}) = P(\text{second} \mid \text{first}) \times P(\text{first choice})$

Step 2: Calculate the probability of selecting a yellow ball on the second draw with a green ball on the first draw

$$P(Y|G) = \frac{P(Green\ AND\ Yellow)}{P(Green)}$$

$$P(Y|G) = \frac{14/39}{4/9}$$

$$P(Y|G) = \frac{14}{39} \times \frac{9}{4}$$

$$P(Y|G) = \frac{126}{156}$$

$$P(Y|G) = \frac{21}{26}$$

Step 3: Write your conclusion: Therefore the probability of selecting a yellow ball on the second draw after drawing a green ball on the first draw is $\frac{21}{26}$.

Example : Two cards are chosen from a deck of cards. What is the probability that they both will be face cards? (draw without replacement)

Solution

Let A = 1st Face card chosen

Let B = 2nd Face card chosen

4 suits → 3 face cards per suit

Therefore, the total number of face cards in the deck = 4 × 3 = 12

$$P(A) = \frac{12}{52}$$

$$P(B) = \frac{11}{51}$$

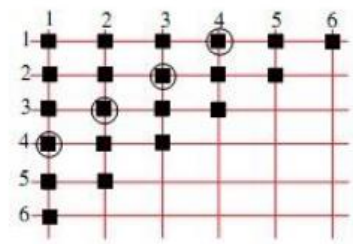
$$P(A\ AND\ B) = \frac{12}{52} \times \frac{11}{51} \text{ or } P(A \cap B) = \frac{12}{52} \times \frac{11}{51} = \frac{33}{663}$$

$$P(A \cap B) = \frac{11}{221}$$

EXAMPLE Two dice numbered one to six are rolled onto a table. Find the probability of obtaining a sum of five given that the sum is seven or less.

Solution We first draw a lattice diagram:

From the diagram we see that the new sample space is made up of 21 outcomes (black boxes) and the event we want (circled) consists of 4 outcomes.



$$P(A \cap B) = P((X=5) \cap (X \leq 7)) = \frac{4}{36}$$

$$P(B) = P(X \leq 7) = \frac{21}{36}.$$

$$P(A \cap B) = P(A|B) \times P(B)$$

$$P(A|B) = \frac{P(B)}{P(A \cap B)}$$

Therefore, $P(X=5|X \leq 7) = \dfrac{\frac{4}{36}}{\frac{21}{36}} = \dfrac{4}{21}.$

EXAMPLE

A box contains 2 red cubes and 4 black cubes. If two cubes are chosen at random, find the probability that both cubes are red given that

(a) the first cube is not replaced before the second cube is selected.

(b) the first cube is replaced before the second cube is selected.

Solution

Let A be the event "the first cube is red" and B be the event "the second cube is red". This means that the event $A \cap B$ must be "both cubes are red".

Now, $p(A) = \frac{2}{6} = \frac{1}{3}$ (as there are 2 red cubes from a total of 6 cubes in the box).

The value of $P(B)$ depends on whether the selection is carried out with or without replacement.

(a) If the first cube selected is red and it is not replaced, then we only have 1 red cube left (in the box) out of a total of five cubes.

So, the probability that the second cube is red given that the first is red is $\frac{1}{5}$.

That is $p(B|A) = \frac{1}{5} \Rightarrow P(A \cap B) = P(B|A) \times P(A) = \frac{1}{5} \times \frac{1}{3} = \frac{1}{15}$.

(b) This time, because the cube is replaced, the probability that the second cube is red given that the first one is red is still $\frac{1}{3}$.

So that, $P(B|A) = \frac{1}{3} \Rightarrow P(A \cap B) = P(B|A) \times P(A) = \frac{1}{3} \times \frac{1}{3} = \frac{1}{9}$.

EXAMPLE Two fair dice are rolled. Find the probability that two even numbers will show up.

Solution

Let the E_1 and E_2 denote the events "An even number on the first die." and "An even number on the second die." respectively.

In this case, the events are physically independent, i.e., the outcome on one die will not influence the outcome on the other die, and so we can confidently say that E_1 and E_2 are independent events.

Therefore, we have $P(E_1 \text{ and } E_2) = P(E_1 \cap E_2) = P(E_1) \times P(E_2) = \frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

Example

A fair six-sided die is tossed twice. What is the probability that a five will occur at least once?

First Solution :

Pr [at least one 5 in two tosses] = Pr [(5 on the first toss) ∪ (5 on the second toss)]

= Pr [5 on the first toss] + Pr [5 on the second toss]

– Pr [(5 on the first toss) ∩ (5 on the second toss)]

$$=\frac{1}{6}+\frac{1}{6}-\left(\frac{1}{6}\right)\left(\frac{1}{6}\right)=\frac{6}{36}+\frac{6}{36}-\frac{1}{36}=\frac{11}{36}$$

Second solution (and the fastest): The probability of no fives in two tosses is

$$\left(\frac{5}{6}\right)\left(\frac{5}{6}\right)=\frac{25}{36}$$

Because the only alternative to no fives is at least one five,

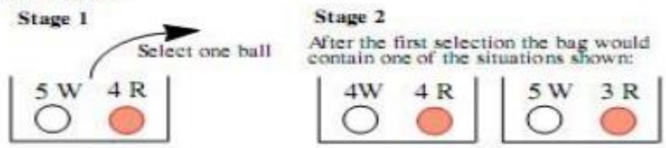
$$\text{Pr [at least one 5 in two tosses]} = 1 - \frac{25}{36}=\frac{11}{36}$$

EXAMPLE A bag contains 5 white balls and 4 red balls. Two balls are selected in such a way that the first ball drawn is not replaced before the next ball is drawn. Find the probability of selecting exactly one white ball.

Solution

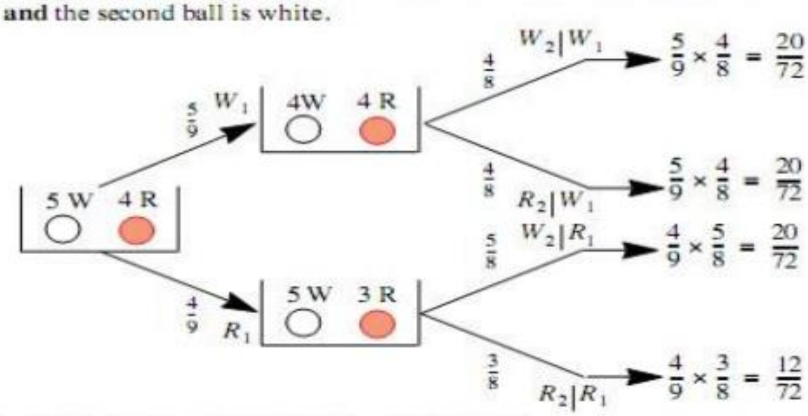
We begin by drawing a diagram of the situation:

From our diagram we notice that there are two possible sample spaces for the second selection.



As an aid, we make use of a tree diagram, where W_i denotes the event "A white ball is selected on the ith trial" and R_i denotes the event "A red ball is selected on the ith trial".

The event "Only one white" occurs if the first ball is white **and** the second ball is red, **or** the first ball is red **and** the second ball is white.



$$
\begin{aligned}
\text{P(One White ball)} &= P(W_1 \cap R_2) + P(R_1 \cap W_2) \\
&= P(R_2|W_1) \times P(W_1) + P(W_2|R_1) \times P(R_1) \\
&= \frac{4}{8} \times \frac{5}{9} + \frac{5}{8} \times \frac{4}{9} \\
&= \frac{5}{9}
\end{aligned}
$$