Chapter three
Derivatives

Let $y = f(x)$ be a function of x . If the limit :

$$\frac{dy}{dx} = f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

exists and is finite , we call this limit the derivative of f at x and say that f is differentiable at x .

EX-1 – Find the derivative of the function : $f(x) = \dfrac{1}{\sqrt{2x+3}}$

***Sol.*:**

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\dfrac{1}{\sqrt{2(x+\Delta x)+3}} - \dfrac{1}{\sqrt{2x+3}}}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{\sqrt{2x+3} - \sqrt{2(x+\Delta x)+3}}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}} . \frac{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}{\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3}}$$

$$= \lim_{\Delta x \to 0} \frac{(2x+3) - (2(x+\Delta x)+3)}{\Delta x.\sqrt{2(x+\Delta x)+3}\sqrt{2x+3}(\sqrt{2x+3} + \sqrt{2(x+\Delta x)+3})}$$

$$= \frac{-2}{(2x+3)(\sqrt{2x+3} + \sqrt{2x+3})} = -\frac{1}{\sqrt{(2x+3)^3}}$$

Rules of derivatives : Let c and n are constants, u , v and w are differentiable functions of x :

1. $\dfrac{d}{dx} c = 0$
2. $\dfrac{d}{dx} u^n = n u^{n-1} \dfrac{du}{dx} \Rightarrow \dfrac{d}{dx}\left(\dfrac{1}{u}\right) = -\dfrac{1}{u^2}\dfrac{du}{dx}$
3. $\dfrac{d}{dx} cu = c\dfrac{du}{dx}$
4. $\dfrac{d}{dx}(u \mp v) = \dfrac{du}{dx} \mp \dfrac{dv}{dx}$; $\dfrac{d}{dx}(u \mp v \mp w) = \dfrac{du}{dx} \mp \dfrac{dv}{dx} \mp \dfrac{dw}{dx}$
5. $\dfrac{d}{dx}(u.v) = u.\dfrac{dv}{dx} + v\dfrac{du}{dx}$

and $\dfrac{d}{dx}(u.v.w)=u.v\dfrac{dw}{dx}+u.w\dfrac{dv}{dx}+v.w\dfrac{du}{dx}$

6. $\dfrac{d}{dx}\left(\dfrac{u}{v}\right)=\dfrac{v\dfrac{du}{dx}-u\dfrac{dv}{dx}}{v^2}$ where $v\neq 0$

EX-2- Find $\dfrac{dy}{dx}$ for the following functions :

a) $y=(x^2+1)^5$

b) $y=[(5-x)(4-2x)]^2$

c) $y=(2x^3-3x^2+6x)^{-5}$

d) $y=\dfrac{12}{x}-\dfrac{4}{x^3}+\dfrac{3}{x^4}$

e) $y=\dfrac{(x^2+x)(x^2-x+1)}{x^3}$

f) $y=\dfrac{x^2-1}{x^2+x-2}$

Sol.-

a) $\dfrac{dy}{dx}=5(x^2+1)^4.2x=10x(x^2+1)^4$

b) $\dfrac{dy}{dx}=2[(5-x)(4-2x)][-2(5-x)-(4-2x)]$

$=8(5-x)(2-x)(2x-7)$

c) $\dfrac{dy}{dx}=-5(2x^3-3x^2+6x)^{-6}(6x^2-6x+6)$

$=-30(2x^3-3x^2+6x)^{-6}(x^2-x+1)$

d) $y=12x^{-1}-4x^{-3}+3x^{-4}\Rightarrow\dfrac{dy}{dx}=-12x^{-2}+12x^{-4}-12x^{-5}$

$\Rightarrow\dfrac{dy}{dx}=-\dfrac{12}{x^2}+\dfrac{12}{x^4}-\dfrac{12}{x^5}$

e) $y=\dfrac{(x+1)(x^2-x+1)}{x^3}\Rightarrow$

$\dfrac{dy}{dx}=\dfrac{x^3[(x^2-x+1)+(x+1)(2x-1)]-3x^2(x+1)(x^2-x+1)}{x^6}=-\dfrac{3}{x^4}$

f) $\dfrac{dy}{dx}=\dfrac{2x(x^2+x-2)-(x^2-1)(2x+1)}{(x^2+x-2)^2}=\dfrac{x^2-2x+1}{(x^2+x-2)^2}$

The Chain Rule:

1. Suppose that $h = g_o f$ is the composite of the differentiable functions $y = g(t)$ and $x = f(t)$, then h is a differentiable functions of x whose derivative at each value of x is :

$$\frac{dy}{dx} = \frac{dy}{dt} \div \frac{dx}{dt}$$

2. If y is a differentiable function of t and t is differentiable function of x, then y is a differentiable function of x :

$$y = g(t) \ \ and \ \ t = f(x) \Rightarrow \frac{dy}{dx} = \frac{dy}{dt} * \frac{dt}{dx}$$

EX-3 – Use the chain rule to express dy/dx in terms of x and y :

a) $y = \frac{t^2}{t^2+1}$ and $t = \sqrt{2x+1}$

b) $y = \frac{1}{t^2+1}$ and $x = \sqrt{4t+1}$

c) $y = \left(\frac{t-1}{t+1}\right)^2$ and $x = \frac{1}{t^2} - 1$ at $t = 2$

d) $y = 1 - \frac{1}{t}$ and $t = \frac{1}{1-x}$ at $x = 2$

Sol.-

a) $y = \frac{t^2}{t^2+1} \Rightarrow \frac{dy}{dt} = \frac{2t(t^2+1) - 2t\,t^2}{(t^2+1)^2} = \frac{2t}{(t^2+1)^2}$

$t = (2x+1)^{\frac{1}{2}} \Rightarrow \frac{dt}{dx} = \frac{1}{2}.(2x+1)^{-\frac{1}{2}}.2 = \frac{1}{\sqrt{2x+1}}$

$$\frac{dy}{dx} = \frac{dy}{dt}.\frac{dt}{dx} = \frac{2t}{(t^2+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{2\sqrt{2x+1}}{((2x+1)+1)^2}.\frac{1}{\sqrt{2x+1}} = \frac{1}{2(x+1)^2}$$

b) $y=(t^2+1)^{-1} \Rightarrow \frac{dy}{dx}=-2t(t^2+1)^{-2}=-\frac{2t}{(t^2+1)^2}$

$$x=(4t+1)^{\frac{1}{2}} \Rightarrow \frac{dx}{dt}=\frac{1}{2}(4t+1)^{-\frac{1}{2}}.4=\frac{2}{\sqrt{4t+1}}$$

$$\frac{dy}{dx}=\frac{dy}{dt}\div\frac{dx}{dt}=-\frac{2t}{(t^2+1)^2}\div\frac{2}{\sqrt{4t+1}}=-\frac{t\sqrt{4t+1}}{(t^2+1)^2}$$

$$=-\frac{x^2-1}{4}.x\div\frac{1}{y^2}=-\frac{xy^2(x^2-1)}{4}$$

$$\text{where} \quad x=\sqrt{4t+1} \Rightarrow t=\frac{x^2-1}{4}$$

$$\text{where} \quad y=\frac{1}{t^2+1} \Rightarrow t^2+1=\frac{1}{y}$$

c) $y=\left(\frac{t-1}{t+1}\right)^2 \Rightarrow \frac{dy}{dt}=2\left(\frac{t-1}{t+1}\right)\frac{t+1-(t-1)}{(t+1)^2}=\frac{4(t-1)}{(t+1)^3}$

$$\Rightarrow \left[\frac{dy}{dt}\right]_{t=2}=\frac{4(2-1)}{(2+1)^3}=\frac{4}{27}$$

$$x=\frac{1}{t^2}-1 \Rightarrow \frac{dx}{dt}=-\frac{2}{t^3} \Rightarrow \left[\frac{dx}{dt}\right]_{t=2}=-\frac{2}{2^3}=-\frac{1}{4}$$

$$\left[\frac{dy}{dx}\right]_{t=2}=\left[\frac{dy}{dt}\div\frac{dx}{dt}\right]_{t=2}=\frac{4}{27}\div\left(-\frac{1}{4}\right)=-\frac{16}{27}$$

d) $t=\frac{1}{1-x}=\frac{1}{1-2}=-1 \quad at \quad x=2$

$$y=1-\frac{1}{t} \Rightarrow \frac{dy}{dt}=\frac{1}{t^2} \Rightarrow \left[\frac{dy}{dt}\right]_{t=-1}=\frac{1}{(-1)^2}=1$$

$$t=(1-x)^{-1} \Rightarrow \frac{dt}{dx}=-(1-x)^{-2}(-1)=\frac{1}{(1-x)^2}$$

$$\Rightarrow \left[\frac{dt}{dx}\right]_{x=2}=\frac{1}{(1-2)^2}=1$$

$$\left[\frac{dy}{dx}\right]_{x=2}=\left[\frac{dy}{dt}\right]_{x=2}.\left[\frac{dt}{dx}\right]_{x=2}=1*1=1$$

Higher derivatives **: If a function $y=f(x)$ possesses a derivative at every point of some interval, we may form the function $f'(x)$ and talk**

about its derivate , if it has one . The procedure is formally identical with that used before , that is :

$$\frac{d^2y}{dx^2}=\frac{d}{dx}\left(\frac{dy}{dx}\right)=\frac{d}{dx}f'(x)=\lim_{\Delta x\to 0}\frac{f'(x+\Delta x)-f'(x)}{\Delta x}$$

if the limit exists .

This derivative is called the second derivative of y with respect to x . It is written in a number of ways , for example,

$$y'' ,\ f''(x) ,\ or\ \frac{d^2 f(x)}{dx^2} .$$

In the same manner we may define third and higher derivatives , using similar notations . The *nth* derivative may be written :

$$y^{(n)} , f^{(n)}(x) , \frac{d^n y}{dx^n} .$$

EX-4- **Find all derivatives of the following function :**

$$y = 3x^3 - 4x^2 + 7x + 10$$

Sol.-

$$\frac{dy}{dx}=9x^2-8x+7 \quad , \quad \frac{d^2y}{dx^2}=18x-8$$

$$\frac{d^3y}{dx^3}=18 \quad , \quad \frac{d^4y}{dx^4}=0=\frac{d^5y}{dx^5}=....$$

Ex-5 **– Find the third derivative of the following function :**

$$y=\frac{1}{x}+\sqrt{x^3}$$

Sol.-

$$\frac{dy}{dx}=-\frac{1}{x^2}+\frac{3}{2}x^{\frac{1}{2}}$$

$$\frac{d^2y}{dx^2}=\frac{2}{x^3}+\frac{3}{4}x^{-\frac{1}{2}}$$

$$\frac{d^3y}{dx^3}=-\frac{6}{x^4}-\frac{3}{8}x^{-\frac{3}{2}} \quad \Rightarrow \frac{d^3y}{dx^3}=-\frac{6}{x^4}-\frac{3}{8\sqrt{x^3}}$$

Implicit Differentiation: **If the formula for f is an algebraic combination of powers of x and y. To calculate the derivatives of these implicitly defined functions, we simply differentiate both sides of the defining equation with respect to x.**

EX-6- **Find $\frac{dy}{dx}$ for the following functions:**

a) $x^2 . y^2 = x^2 + y^2$

b) $(x+y)^3 + (x-y)^3 = x^4 + y^4$

c) $\frac{x-y}{x-2y} = 2$ at $P(3,1)$

d) $xy + 2x - 5y = 2$ at $P(3,2)$

Sol.

a) $x^2(2y\frac{dy}{dx}) + y^2(2x) = 2x + 2y\frac{dy}{dx} \Rightarrow \frac{dy}{dx} = \frac{x - xy^2}{x^2 y - y}$

b) $3(x+y)^2(1+\frac{dy}{dx}) + 3(x-y)^2(1-\frac{dy}{dx}) = 4x^3 + 4y^3\frac{dy}{dx}$

$$\Rightarrow \frac{dy}{dx} = \frac{4x^3 - 3(x+y)^2 - 3(x-y)^2}{3(x+y)^2 - 3(x-y)^2 - 4y^3} \Rightarrow \frac{dy}{dx} = \frac{2x^3 - 3x^2 - 3y^2}{6xy - 2y^3}$$

c) $\frac{(x-2y)(1-\frac{dy}{dx}) - (x-y)(1-2\frac{dy}{dx})}{(x-2y)^2} = 0 \Rightarrow \frac{dy}{dx} = \frac{y}{x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,1)} = \frac{1}{3}$

d) $x\frac{dy}{dx} + y + 2 - 5\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y+2}{5-x} \Rightarrow \left[\frac{dy}{dx}\right]_{(3,2)} = \frac{2+2}{5-3} = 2$

Exponential functions : **If u is any differentiable function of x, then :**

7) $\frac{d}{dx}a^u = a^u . \ln a . \frac{du}{dx}$ and $\frac{d}{dx}e^u = e^u . \frac{du}{dx}$

EX-7 –Find $\frac{dy}{dx}$ for the following functions :

a) $y = 2^{3x}$ b) $y = 2^x . 3^x$

c) $y = (2^x)^2$ d) $y = x.2^{x^2}$

e) $y = e^{(x + e^{5x})}$ f) $y = e^{\sqrt{1+5x^2}}$

Sol.-

a) $y = 2^{3x} \Rightarrow \frac{dy}{dx} = 2^{3x} * 3 \ln 2$

b) $y = 2^x . 3^x \Rightarrow y = 6^x \Rightarrow \frac{dy}{dx} = 6^x . \ln 6$

c) $y = (2^x)^2 \Rightarrow y = 2^{2x} \Rightarrow \frac{dy}{dx} = 2^{2x} \ln 2 . 2 = 2^{2x+1} \ln 2$

d) $y = x.2^{x^2} \Rightarrow \frac{dy}{dx} = x.2^{x^2} \ln 2 . 2x + 2^{x^2} = 2^{x^2}(2x^2 \ln 2 + 1)$

e) $y = e^{(x + e^{5x})} \Rightarrow \frac{dy}{dx} = e^{(x + e^{5x})}(1 + 5e^{5x})$

f) $y = e^{(1+5x^2)^{\frac{1}{2}}} \Rightarrow \frac{dy}{dx} = e^{(1+5x^2)^{\frac{1}{2}}} \frac{1}{2}(1+5x^2)^{-\frac{1}{2}} . 10x = e^{\sqrt{1+5x^2}} \frac{5x}{\sqrt{1+5x^2}}$

Logarithm functions : If u is any differentiable function of x , then :

$$8)\ \frac{d}{dx} \log_a u = \frac{1}{u . \ln a} . \frac{du}{dx} \quad \text{and} \quad \frac{d}{dx} \ln u = \frac{1}{u} . \frac{du}{dx}$$

EX-8 – Find $\frac{dy}{dx}$ for the following functions :

a) $y = \log_{10} e^x$ b) $y = \log_5 (x+1)^2$

c) $y = \log_2 (3x^2 + 1)^3$ d) $y = [\ln(x^2 + 2)^2]^3$

e) $y + \ln(xy) = 1$ f) $y = \frac{(2x^3 - 4)^{\frac{2}{3}} . (2x^2 + 3)^{\frac{5}{2}}}{(7x^3 + 4x - 3)^2}$

Sol. –

$$a)\ y=\log_{10}e^{x} \Rightarrow y = x\log_{10}e \Rightarrow \frac{dy}{dx}=\log_{10}e=\frac{\ln e}{\ln 10}=\frac{1}{\ln 10}$$

$$b)\ y=\log_{5}(x+1)^{2}=2\log_{5}(x+1) \Rightarrow \frac{dy}{dx}=\frac{2}{(x+1)\ln 5}$$

$$c)\ y=3\log_{2}(3x^{2}+1) \Rightarrow \frac{dy}{dx}=\frac{3}{3x^{2}+1}.\frac{6x}{\ln 2}=\frac{18x}{(3x^{2}+1)\ln 2}$$

$$d)\ \frac{dy}{dx}=3\left[2\ln(x^{2}+2)\right]^{2}\frac{2}{x^{2}+2}.2x=\frac{48x\left[\ln(x^{2}+2)\right]^{2}}{x^{2}+2}$$

$$e)\ y+\ln x+\ln y=1 \Rightarrow \frac{dy}{dx}+\frac{1}{x}+\frac{1}{y}.\frac{dy}{dx}=0 \Rightarrow \frac{dy}{dx}=-\frac{y}{x(y+1)}$$

$$f)\ \ln y=\frac{2}{3}\ln(2x^{3}-4)+\frac{5}{2}\ln(2x^{2}+3)-2\ln(7x^{3}+4x-3)$$

$$\Rightarrow \frac{1}{y}.\frac{dy}{dx}=\frac{2}{3}.\frac{6x^{2}}{2x^{3}-4}+\frac{5}{2}.\frac{4x}{2x^{2}+3}-2.\frac{21x^{2}+4}{7x^{3}+4x-3}$$

$$\Rightarrow \frac{dy}{dx}=2y\left[\frac{2x^{2}}{2x^{3}-4}+\frac{5x}{2x^{2}+3}-\frac{21x^{2}+4}{7x^{3}+4x-3}\right]$$

Trigonometric functions : If u is any differentiable function of x , then :

$$9)\ \frac{d}{dx}\sin u=\cos u.\frac{du}{dx}$$

$$10)\ \frac{d}{dx}\cos u=-\sin u.\frac{du}{dx}$$

$$11)\ \frac{d}{dx}\tan u=\sec^{2}u.\frac{du}{dx}$$

$$12)\ \frac{d}{dx}\cot u=-\csc^{2}u.\frac{du}{dx}$$

$$13)\ \frac{d}{dx}\sec u=\sec u.\tan u.\frac{du}{dx}$$

$$14)\ \frac{d}{dx}\csc u=-\csc u.\cot u.\frac{du}{dx}$$

_EX-9-_ Find $\frac{dy}{dx}$ for the following functions :

a) $y=\tan(3x^2)$ b) $y=(\csc x+\cot x)^2$

c) $y=2\sin\frac{x}{2}-x\cos\frac{x}{2}$ d) $y=\tan^2(\cos x)$

e) $x+\tan(xy)=0$ f) $y=\sec^4 x-\tan^4 x$

Sol.-

a) $\frac{dy}{dx}=\sec^2(3x^2).6x=6x.\sec^2(3x^2)$

b) $\frac{dy}{dx}=2(\csc x+\cot x)(-\csc x.\cot x-\csc^2 x)=-2\csc x.(\csc x+\cot x)^2$

c) $\frac{dy}{dx}=2\cos\frac{x}{2}.\frac{1}{2}-\left[x(-\sin\frac{x}{2}).\frac{1}{2}+\cos\frac{x}{2}\right]=\frac{x}{2}.\sin\frac{x}{2}$

d) $\frac{dy}{dx}=2.\tan(\cos x).\sec^2(\cos x).(-\sin x)=-2.\sin x.\tan(\cos x).\sec^2(\cos x)$

e) $1+\sec^2(xy).(x\frac{dy}{dx}+y)=0\Rightarrow\frac{dy}{dx}=-\frac{1+y.\sec^2(xy)}{x.\sec^2(xy)}=-\frac{\cos^2(xy)+y}{x}$

f) $\frac{dy}{dx}=4\sec^3 x.\sec x.\tan x-4.\tan^3 x.\sec^2 x=4\tan x.\sec^2 x$

EX-10- Prove that :

a) $\frac{d}{dx}\tan u=\sec^2 u.\frac{du}{dx}$ b) $\frac{d}{dx}\sec u=\sec u.\tan u.\frac{du}{dx}$

Proof :

a) $$L.H.S.=\frac{d}{dx}\tan u=\frac{d}{dx}\frac{\sin u}{\cos u}=\frac{\cos u.\cos u.\frac{du}{dx}-\sin u.(-\sin u)\frac{du}{dx}}{\cos^2 u}$$

$$=\frac{\cos^2 u+\sin^2 u}{\cos^2 u}.\frac{du}{dx}=\frac{1}{\cos^2 u}.\frac{du}{dx}=\sec^2 u.\frac{du}{dx}=R.H.S.$$

b) $$L.H.S.=\frac{d}{dx}\sec u=\frac{d}{dx}\frac{1}{\cos u}=-\frac{1}{\cos^2 u}(-\sin u)\frac{du}{dx}$$

$$=\frac{1}{\cos u}.\frac{\sin u}{\cos u}.\frac{du}{dx}=\sec u.\tan u.\frac{du}{dx}=R.H.S.$$

The inverse trigonometric functions **:** **If *u* is any differentiable function**