Al-Mustaqbal University
Computer Techniques Eng. Dept.
1st Stage Mathematics I
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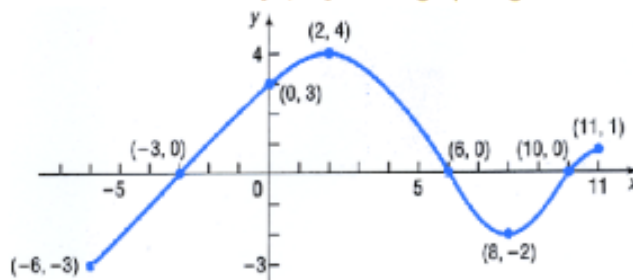1st term – Functions

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Limits: Graphical Solutions

Graphical Limits

Let $f(x)$ be a function defined on the interval [-6,11] whose graph is given as:

The limits are defined as the value that the function approaches as it goes to an x value. Using this definition, it is possible to find the value of the limits given a graph. A few examples are below:

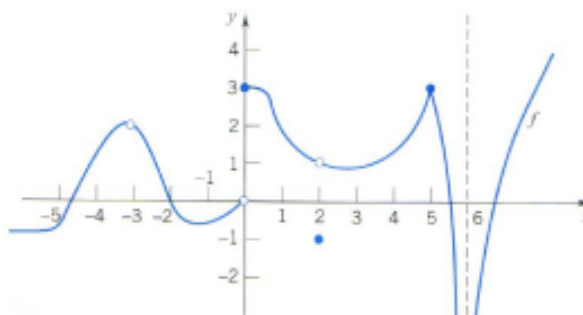$$\lim_{x \to -3} f(x) = 0 \qquad\qquad \lim_{x \to 2} f(x) = 4 \qquad\qquad \lim_{x \to 8} f(x) = -2$$

In general, you can see that these limits are equal to the value of the function. This is true if the function is <u>continuous</u>.

Continuity

Continuity of a graph is loosely defined as the ability to draw a graph without having to lift your pencil. To better understand this, see the graph below:

Let's investigate at the flowing points:

$x = -3$ Discontinuous at this point *The value is not defined at -3* "Removable discontinuity"	$x = 0$ Discontinuous at this point *The limit of the left is not equal to the limit from the right* "Jump discontinuity"	$x = 2$ Discontinuous at this point *The limit from the left is equal to the right, but is not equal to the value of the function* "Removable discontinuity"
$x = 4$ Continuous at this point *The limit from the left is equal to the limit from the right and equal to the value of the function*	$x = 5$ Continuous at this point *The limit from the left is equal to the limit from the right and equal to the value of the function*	$x = 6$ Discontinuous at this point *The value of the limit is equal to negative infinity and therefore not defined* "Infinite discontinuity"

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One-Sided Limits: General Definition

One-sided limits are differentiated as *right-hand limits* (when the limit approaches from the right) and *left-hand limits* (when the limit approaches from the left) whereas ordinary limits are sometimes referred to as *two-sided limits*. Right-hand limits approach the specified point from positive infinity. Left-hand limits approach this point from negative infinity.

The right-handed limit:

$$\lim_{x \to a^+} f(x) = L$$

The left-handed limit:

$$\lim_{x \to a^-} f(x) = L$$

A More Formal Definition of Continuity

From this information, a more formal definition can be found. Continuity, at a point *a*, is defined when the limit of the function from the left equals the limit from the right and this value is also equal to the value of the function. Using notation, for all points *a* where

$$lim_{x \to a^+} f(x) = lim_{x \to a^-} f(x) = f(a),$$

the function is said to be **continuous**.

A. Now you try some!
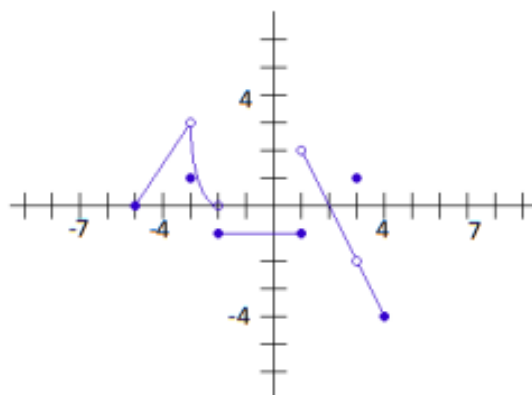
Determine if the following limits exists:

$$\lim_{x \to -3} f(x) \qquad \lim_{x \to -2} f(x) \qquad \lim_{x \to 0} f(x) \qquad \lim_{x \to 1} f(x) \qquad \lim_{x \to 2} f(x) \qquad \lim_{x \to 3} f(x)$$

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Summary: When does a limit not exist?
- A general limit does not exist if the left-and right-hand limits aren't equal (resulting in a discontinuity in the function).
- A general limit does not exist wherever a function increases or decreases infinitely ("without bound") as it approaches a given x-value.
- A general limit does not exist in the cases of infinite oscillation when approaching a fixed point.

Limits: Numeric Solutions

Now that you know how to solve a limit graphically, you may be asking yourself: 'That's great, but what about when there isn't a graph in the problem?' That is a good question, and that is what this next section is about. There are a many better (and more accurate) ways to find the value of the limit than graphing or plugging in numbers that get closer and closer to the value of interest. These solution methods fall under three categories: substitution, factoring, and the conjugate method. But first things first, let's discuss some of the general rules for limits.

Limit Rules

Here are some of the general limit rules (with $\lim_{x \to c} f(x) = L$ and $\lim_{x \to c} g(x) = M$):

1. **Sum Rule:** The limit of the sum of two functions is the sum of their limits
$$\lim_{x \to c}\big(f(x) + g(x)\big) = \lim_{x \to c} f(x) + \lim_{x \to c} g(x) = L + M$$

2. **Difference Rule:** The limit of the difference of two functions is the difference of their limits
$$\lim_{x \to c}\big(f(x) - g(x)\big) = \lim_{x \to c} f(x) - \lim_{x \to c} g(x) = L - M$$

3. **Product Rule:** The limit of a product of two functions is the product of their limits
$$\lim_{x \to c}\big(f(x) \ast g(x)\big) = \lim_{x \to c} f(x) \ast \lim_{x \to c} g(x) = L \ast M$$

4. **Constant Multiple Rule:** The limits of a constant times a function is the constant times the limit of the function
$$\lim_{x \to c}\big(k \ast f(x)\big) = k \lim_{x \to c} f(x) = k \ast L$$

5. **Quotient Rule:** The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero
$$\lim_{x \to c}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x \to c} f(x)}{\lim_{x \to c} g(x)} = \frac{L}{M}, \quad M \neq 0$$

Limit Rule Examples Find the following limits using the above limit rules:

1. $\lim_{x \to c}(x^3 + 4x^2 - 3) = \lim_{x \to c}(x^3) + \lim_{x \to c}(4x^2) - \lim_{x \to c}(3)$
$$= c^3 + 4c^2 - 3$$

2. $\lim_{x \to c}\frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to c}(x^4 + x^2 - 1)}{\lim_{x \to c}(x^2 + 5)}$
$$= \frac{\lim_{x \to c}(x^4) + \lim_{x \to c}(x^2) - \lim_{x \to c}(1)}{\lim_{x \to c}(x^2) + \lim_{x \to c}(5)}$$
$$= \frac{c^4 + c^2 - 1}{c^2 + 5}$$

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3. $\lim_{x \to 1} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \to 1}(x^4 + x^2 - 1)}{\lim_{x \to 1}(x^2 + 5)}$

$$= \frac{\lim_{x \to 1}(x^4) + \lim_{x \to 1}(x^2) - \lim_{x \to 1}(1)}{\lim_{x \to 1}(x^2) + \lim_{x \to 1}(5)}$$

$$= \frac{1^4 + 1^2 - 1}{1^2 + 5} = \frac{1}{6}$$

B. Now you try some!

1. $\lim_{x \to c} (2x + 5)$
2. $\lim_{t \to 6} 8(t - 5)(t - 7)$
3. $\lim_{x \to 2} \frac{x + 2}{x^2 + 5x + 6}$

Limits of Rational Functions: Substitution Method

A rational function is a function that can be written as the ratio of two algebraic expressions. If a function is considered rational and the denominator is not zero, the limit can be found by substitution. This can be seen in the example below (which is similar to the example #3 above, but now done in one quick, convenient step):

$$\lim_{x \to -1} \frac{x^3 + 4x^2 - 3}{x^2 + 5} = \frac{(-1)^3 + 4(-1)^2 - 3}{(-1)^2 + 5} = \frac{0}{6} = 0$$

This can be defined more formally as: If $P(x)$ and $Q(x)$ are algebraic expressions and $Q(c) \neq 0$, then:

$$\lim_{x \to c} \frac{P(x)}{Q(x)} = \frac{P(c)}{Q(c)}$$

C. Now you try some!

1. $\lim_{x \to -5} \frac{x^2 + 3x - 5}{x + 7}$
2. $\lim_{x \to 2} \frac{x + 3}{x + 6}$

Factoring Method

Consider the function $(x) = \frac{x^2 - 9}{x + 3}$. How would you find the limit of f as x approaches -3? If you try to use substitution to find the limit, world-ending paradoxes ensue:

$$\lim_{x \to -3} \frac{x^2 - 9}{x + 3} = \frac{(-3)^2 - 9}{(-3) + 3} = \frac{0}{0}$$

But fear not, this answer just tells us that we must use a different method to find the limit, because the function likely has a "hole" at the given x value. Therefore, the factoring method can be tried. To start this method, the numerator and denominator must be factored (in this case the denominator is "factored" already).

$$\lim_{x \to -3} \frac{(x + 3)(x - 3)}{x + 3}$$

The factor $(x + 3)$ can be canceled to get the much simpler limit expression of $\lim_{x \to -3}(x - 3)$ that can easily be evaluated via substitution:

$$\lim_{x \to -3} (x - 3) = -3 - 3 = -6$$

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Therefore, the result of the limit can be found, with the understanding that there is a "hole" in the graph at $x = -3$. Hence, $\lim_{x \to -3} \frac{x^2-9}{x+3} = -6$.

D. Now you try some!

a) $\lim_{x \to 5} \frac{2x^2-7x-15}{x-5}$

b) $\lim_{x \to 1} \frac{x^3-1}{x-1}$

Conjugate Method

The conjugate of a binomial expression (i.e. an expression with two terms, *you can tell this because of the Latin root bi- meaning two*) is the same expression with opposite middle signs. For example, the conjugate of $(\sqrt{x} - 5)$ is $(\sqrt{x} + 5)$. This is really useful if you have a radical in your limit. This is because the product of two conjugates containing radicals will, itself, contain no radical expressions. See below:

$$(\sqrt{x} - 5)(\sqrt{x} + 5) = \sqrt{x^2} + 5\sqrt{x} - 5\sqrt{x} - 25 = x - 25$$

You should use the conjugate method whenever you have a limit problem containing radicals for which substitution does not work.

Example:

Evaluate $\lim_{x \to 5} \frac{\sqrt{x+11}-4}{x-5}$

First try the substitution method:

$$\lim_{x \to 5} \frac{\sqrt{x+11}-4}{x-5} = \frac{\sqrt{5+11}-4}{5-5} = \frac{0}{0}$$

Well, another hole in the universe, or at least the graph. Indicating that you'll need another method to find the limit since the function probably has a hole at $x = 5$. To start, multiply both the numerator and denominator by the conjugate of the radical expression $(\sqrt{x+11}+4)$:

$$\lim_{x \to 5} \frac{\sqrt{x+11}-4}{x-5} * \frac{(\sqrt{x+11}+4)}{(\sqrt{x+11}+4)}$$

$$\lim_{x \to 5} \frac{(x+11)-16}{(x-5)(\sqrt{x+11}+4)}$$

$$\lim_{x \to 5} \frac{x-5}{(x-5)(\sqrt{x+11}+4)}$$

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Cancel the $x - 5$ factor in the numerator and denominator.

$$\lim_{x \to 5} \frac{1}{\left(\sqrt{x+11}+4\right)} = \frac{1}{\left(\sqrt{5+11}+4\right)} = \frac{1}{8}$$

E. **Now you try some!**

a) $\lim_{x \to -2} \frac{x+2}{\sqrt{x+6}-2}$

b) $\lim_{x \to -1} \frac{\sqrt{x+10}-3}{x+1}$

Formal Definition: Limits

Limits are more formally defined as

"L is the limit of f(x) as x approaches a if for every number $\in > 0$, there is a corresponding number $\delta > 0$ such that for all x. Using notation we write

$$\lim_{x \to a} f(x) = L \quad \text{IFF} \quad 0 < |x - a| < \delta \;\; \to \;\; |f(x) - L| < \in$$

From the formal definition, right-handed limit can be defined as:

$$0 < x - a < \delta \;\; \to \;\; |f(x) - L| < \in$$

And written as:

$$\lim_{x \to a^+} f(x) = L$$

Whereas the left-handed limit can be defined as:

$$-\delta < x - a < 0 \;\; \to \;\; |f(x) - L| < \in$$

And written as:

$$\lim_{x \to a^-} f(x) = L$$

Example 1: Testing the Definition

Show: $\lim_{x \to 1} (5x - 3) = 2$

We have to find a suitable $\delta > 0$ so that if $x \neq 1$ and x is within distance δ, that is if:

$$0 < |x - 1| < \delta$$

Then $f(x)$ is within distance $\in$ of $L = 2$, that is

$$|f(x) - 2| < \in$$
$$|(5x - 3) - 2| < \in$$
$$|5x - 5| < \in$$
$$5|x - 1| < \in$$
$$|x - 1| < \frac{\in}{5}$$

Thus, we can take $\delta = \frac{\in}{5}$ due to the fact that $0 < |x - 1| < \delta$, then:

$$|f(x) - 2| < \in$$

So, if $|(5x - 3) - 2| = |5x - 5| = 5|x - 1| < 5(\delta) = 5\left(\frac{\epsilon}{5}\right) = \epsilon$

therefore, $0 < |x - 1| < \frac{\epsilon}{5} \rightarrow |f(x) - 2| < \epsilon$ and $\lim_{x \to 1}(5x - 3) = 2$.

Limits: Advanced Topics

Previously, when we found that the result of a limit doing straight substitution yielded $\frac{0}{0}$ we used factoring or conjugation to be able to solve the problem. What happens when neither of those methods prove useful? You become very grateful for the 17th-century French mathematician Guillaume de L'Hôpital. L'Hôpital was the man that derived a method of solving these types of equations, known as indeterminate forms. This method, known as L'Hôpital's Rule, is formally defined below.

Formal Definition: L'Hôpital's Rule

If the limit $\lim_{x \to c} \frac{f(x)}{g(x)}$ results in one of the following forms:

$$\frac{0}{0}, \qquad \pm\frac{\infty}{\infty}, \qquad 0 * \pm\infty, \qquad \infty - \infty, \qquad 0^0, \qquad 1^\infty, \qquad \infty^0$$

And $\lim_{x \to c} \frac{f'(x)}{g'(x)}$ exits and $g'(x) \neq 0$, then:

$$\lim_{x \to c} \frac{f(x)}{g(x)} = \lim_{x \to c} \frac{f'(x)}{g'(x)}$$

Example 1: indeterminate form of $\frac{0}{0}$

Find the limit $\lim_{x \to 0} \frac{e^x - 1}{x}$

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \frac{0}{0} = \lim_{x \to 0} \frac{e^x}{1}$$

Using L'Hôpital's Rule:

$$\lim_{x \to 0} \frac{e^x - 1}{x} = \lim_{x \to 0} \frac{e^x}{1} = \frac{1}{1} = 1$$

Example 2: indeterminate form of $\frac{\infty}{\infty}$

Find the limit $\lim_{x \to \infty} \frac{x^2}{2^x}$

$$\lim_{x \to \infty} \frac{x^2}{2^x} = \frac{\infty}{\infty}$$

Using L'Hôpital's Rule:

$$\lim_{x \to \infty} \frac{x^2}{2^x} = \lim_{x \to \infty} \frac{2x}{2^x ln2} = \frac{2}{ln2} \lim_{x \to \infty} \frac{x}{2^x} = \frac{\infty}{\infty}$$

Using L'Hôpital's Rule again:

$$\frac{2}{ln2} \lim_{x \to \infty} \frac{x}{2^x} = \frac{2}{ln2} \lim_{x \to \infty} \frac{1}{2^x ln2} = \frac{2}{(ln2)^2} \lim_{x \to \infty} \frac{1}{2^x} = \frac{2}{(ln2)^2} * 0 = 0$$

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Example 3: indeterminate form of $\infty - \infty$

Find the limit $\lim_{x \to 2} \left(\frac{4}{x^2-4} - \frac{1}{x-2} \right)$

$$\lim_{x \to 2} \left(\frac{4}{x^2 - 4} - \frac{1}{x - 2} \right) = \infty - \infty$$

Using L'Hôpital's Rule:

$$\lim_{x \to 2} \left(\frac{4}{x^2 - 4} - \frac{1}{x - 2} \right) = \lim_{x \to 2} \frac{4 - (x + 2)}{(x - 2)(x + 2)} = \lim_{x \to 2} \frac{2 - x}{x^2 - 4} = \lim_{x \to 2} \frac{-1}{2x} = -\frac{1}{4}$$

Example 4: indeterminate form of 1^∞

Find the limit $\lim_{x \to 1} x^{\frac{1}{1-x}}$

$$\lim_{x \to 1} x^{\frac{1}{1-x}} = 1^\infty$$

Let $y = x^{\frac{1}{1-x}}$. Then $\ln y = \ln x^{\frac{1}{1-x}} = \frac{\ln x}{1-x}$

Using L'Hôpital's Rule:

$$\lim_{x \to 1} \ln y = \lim_{x \to 1} \frac{\ln x}{1 - x} = \lim_{x \to 1} \frac{\frac{1}{x}}{-1} = -\lim_{x \to 1} \frac{1}{x} = -1$$

Therefore

$$\lim_{x \to 1} \ln y = -1$$

$$\lim_{x \to 1} x^{\frac{1}{1-x}} = \lim_{x \to 1} y = e^{-1} = \frac{1}{e}$$

Example 5: indeterminate form of $0 * \infty$

Find the limit $\lim_{x \to 0^+} (x * \ln x)$

$$\lim_{x \to 0^+} x * \ln x = 0 * \infty$$

Using L'Hôpital's Rule:

$$\lim_{x \to 0^+} x * \ln x = \lim_{x \to 0^+} \frac{\ln x}{\frac{1}{x}} = \lim_{x \to 0^+} \frac{\frac{1}{x}}{-\frac{1}{x^2}} = \lim_{x \to 0^+} -x = 0$$

F. Now you try some!

1. $\lim_{x \to 0} \frac{\sin 3x}{4x}$

2. $\lim_{x \to 1} \frac{5x^4 - 4x^2 - 1}{10 - x - 9x^3}$

3. $\lim_{x \to \infty} \frac{e^x}{x^3}$

4. $\lim_{x \to -\infty} x * \ln x$

5. $\lim_{x \to \infty} x^{\frac{1}{x}}$

6. Why does the $\lim_{x \to -1} \frac{\sqrt{x+4}-3}{x+1}$ not equal $1/2\sqrt{3}$?

Note the trick that is needed here – what happens if you use L'Hôpital's Rule without making this initial change?

7. Early in their presentation of limits, most teachers present the following two limit facts as special cases: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$ and $\lim_{\theta \to 0} \frac{\cos \theta - 1}{\theta} = 0$. Verify each by using L'Hôpital's Rule.

Limits to Infinity (Horizontal Asymptotes)

What happens to a function as it goes further and further out to the left and right? Well, that depends on the function. But half of the answer can be discovered by allowing the independent variable to take on increasingly large, positive values and keeping an eye on the output (the graph) - this investigates what is happening as we go further and further to the right. The other half is discovered by allowing the independent variable to take on increasingly large, negative values and, again, keeping an eye on the output – this investigates what is happening as we go further and further to the left.

Here are some basic facts and some generalizations that will be sufficient to evaluate most "limits to infinity".

Consider the function as an algebraic fraction, and consider the ratio of the leading terms. Let the algebraic expression in the numerator be expressed as *n(x)* and the algebraic expression found in the denominator be expressed as *d(x)*, then

- If the degree of the numerators is lower than the degree of the denominator, then $\lim_{x \to \infty} \frac{n(x)}{d(x)} = 0$. In general, whenever the denominator grows faster than the numerator, the limit will go to zero. Thus, in these cases, as the graph extends far to both the left and the right, the output (i.e., the graph) gets closer and closer to zero.

  Here is a list of functions in order of their rate of growth – quickest to slowest:
  $x!, \dots 4^x, 3^x, 2^x, \dots x^4, x^3, x^2, x \log x, x, \log x, \dots, 3, 2, 1$

  Examples:
  a. $\lim_{x \to \infty} \frac{5 x^3 - 2x^2 + 7x - 13}{12 - 2x + x^4} = \lim_{x \to \infty} \frac{5x^3}{x^4} = \lim_{x \to \infty} \frac{5}{x} = 0$
  b. $\lim_{x \infty \to} \frac{6x \log x + 5x - 2}{2^x \mp 2x - 14 \log x} = \lim_{x \to \infty} \frac{6x \log x}{2^x} = 0$

- If the degree of the numerator is higher than the degree of the denominator, then $\lim_{x \to \infty} \frac{n(x)}{d(x)} = \infty \ or - \infty$. In general, whenever the numerator grows faster than the denominator, the limit will go to positive or negative infinity. Thus, in these cases, as the graph extends far to both the left and the right, the output (i.e., the graph) increases or decreases without bound. In these cases, each side needs to be considered separately.

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Examples:

a. $\lim_{x \to \infty} \frac{3x^5 + x^2 - 5}{6 + x + 7x^2} = \lim_{x \to \infty} \frac{3x^5}{7x^2} = \lim_{x \to \infty} \frac{3}{7} x^3 = \infty$

whereas $\lim_{x \to -\infty} \frac{3x^5 + x^2 - 5}{6 + x + 7x^2} = \lim_{x \to -\infty} \frac{3}{7} x^3 = -\infty$

b. $\lim_{x \to \infty} \frac{x! - \log x + 1}{2^x + 3x^5 - 5 \log x} = \lim_{x \to \infty} \frac{x!}{2^x} = +\infty$ [Note that $\lim_{x \to -\infty} \frac{x! - \log x + 1}{2^x + 3x^5 - 5 \log x}$

   is meaningless since $x!$ is not defined for negative values.]

c. $\lim_{x \to \infty} 2x^4 - 5x + 3 = +\infty$ (Note that $\lim_{x \to -\infty} 2x^4 - 5x + 3 = +\infty$ as well [why?])

- If the degrees are equal, then $\lim_{x \to \infty} \frac{n(x)}{d(x)}$ is equal to the leading coefficient of $n(x)$ over the leading coefficient of $d(x)$.

Examples:

a. $\lim_{x \to \infty} \frac{3x^3 - 2x^2 + x - 4}{4 + 2x - 5x^3} = \lim_{x \to \infty} \frac{3x^3}{-5x^3} = \lim_{x \to \infty} \frac{3}{-5} = -\frac{3}{5}$

b. $\lim_{x \to \infty} \frac{\frac{1}{2} 2^x}{\frac{3}{4} 2^x - \log x + 4} = \frac{1/2}{3/4} = \frac{2}{3}$

G. Now you try some!

1. $\lim_{x \to -\infty} \frac{2x^3 - 2x + 5}{13x^3 + x^2 - 5x + 13}$

2. $\lim_{x \to +\infty} \frac{5x + 1}{x^2 - 3x + 4}$

3. $\lim_{x \to -\infty} \frac{2^x}{x^2}$

4. $\lim_{x \to +\infty} \frac{2^x}{x^2}$

5. $\lim_{x \to -\infty} -3x^3 + 2x^2 - 4x + 5$