1.1 Overview

Functions are fundamental to the study of calculus. In this chapter, we review what functions are and how they are visualized as graphs, how they are combined and transformed, and ways they can be classified.

1.2 Domain and Range

The temperature at which water boils depends on the elevation above sea level. The interest paid on a cash investment depends on the time the investment is held. The area of a circle depends on the radius of the circle. The distance an object travels depends on the elapsed time. In each case, the value of one variable quantity, say y, depends on the value of another variable amount, which we often call x. We say that "y is a function of x" and write this symbolically as

$$\boldsymbol{y = f(x)} \qquad (\text{"}\mathbf{y\ equals}\ f\ \mathbf{of\ x}\text{"})$$

The symbol f represents the function, the letter x is the independent variable representing the input value to f, and y is the dependent variable or output value of f at x.

Definition A function f from a set D to a set Y is a rule that assigns a unique value f(x) in Y to each x in D.

The set D of all possible input values is called the **domain of the function**. The set of all output values of f(x) as x varies throughout D is called the **range of the function**. The range might not include every element in the set Y. The domain and range of a function can be any set of objects, but often in calculus, they are sets of real numbers interpreted as points of a coordinate line. Often a function is given by a formula that describes how to calculate the output value from the input variable.

For instance, the equation $A = \pi r^2$ is a rule that calculates the area A of a circle from its radius r. When we define a function y = f(x) with a formula and the domain is not stated explicitly or restricted by context, the domain is assumed to be the largest set of real x-values for which the formula gives real y-values. This is called the **natural domain** of f. If we want to restrict the domain in some way, we must say so. The domain of $y = x^2$ is the entire set of real numbers. To restrict the domain of the function to, say, positive values of x, we would write $y = x^2$, $x > 0$. Changing the domain to which we apply a formula usually changes the range as well. The range of is $y = x^2$ is [0, ∞). The range of $y = x^2$, x ≥ 2, is the set of all numbers obtained by squaring numbers greater than or equal to 2.

A function f is like a machine that produces an output value f(x) in its range whenever we feed it an input value x from its domain (Figure 1.1). The function keys on a calculator give an example of a function as a machine.

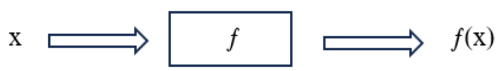


Figure 1-1 A diagram showing a function as a kind of machine.

A function can also be pictured as an arrow diagram (Figure 1.2). Each arrow associates to an element of the domain D a single element in the set Y. In Figure 1.2, the arrows indicate that f(a) is associated with a, f(x) is associated with x, and so on. Notice that a function can have the same output value for two different input elements in the domain (as occurs with f(a) in Figure 1.2), but each input element x is assigned a single output value f(x).

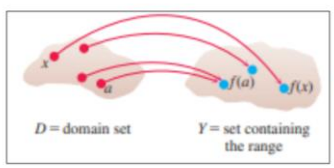


Figure 1-2 A function from a set D to a set Y assigns a unique element of Y to each element in D

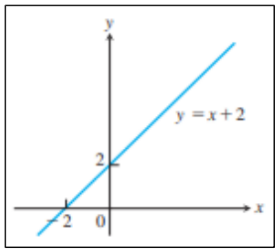
Example 1: Verify the natural domains and associated ranges of some simple functions

1.3 Graphs of Functions

If f is a function with domain D, its graph consists of the points in the Cartesian plane whose coordinates are the input-output pairs for f. In set notation, the graph is:

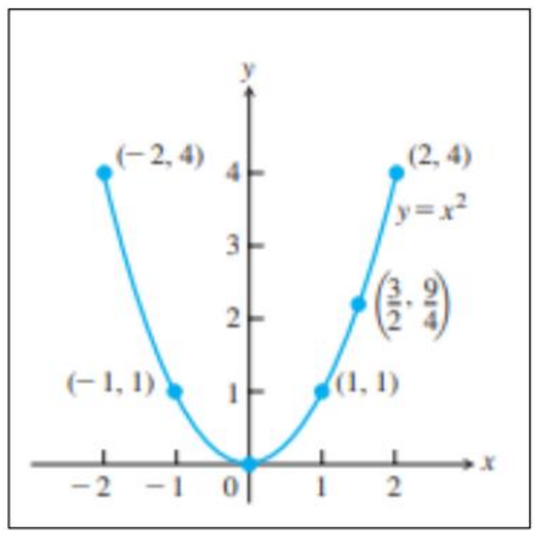
$$\{(\boldsymbol{x}, \boldsymbol{f}(\boldsymbol{x}))|\boldsymbol{x} \in \boldsymbol{D}\}$$

The graph of the function $f(x) = x + 2$ is the set of points with coordinates (x, y) for which $y = x + 2$. Its graph is the straight line sketched in Figure below.



Example 2: Graph the function $y = x^2$ over the interval [−2, 2]

x	$y = x^2$
−2	4
−1	1
0	0
1	1
$\frac{3}{2}$	$\frac{9}{4}$
2	4



1.4 Piecewise-Defined Functions

Sometimes a function is described in pieces by using different formulas on different parts of its domain. One example is the **absolute value function.**

$$|x| = \begin{cases} x, x \geq 0 & \text{first formala} \\ -x, x < 0 & \text{second formala} \end{cases}$$

whose graph is given in Figure 1.3. The right-hand side of the equation means that the function equals x if $x \geq 0$, and equals -x if $x < 0$. Piecewise-defined functions often arise when real world data are modeled.

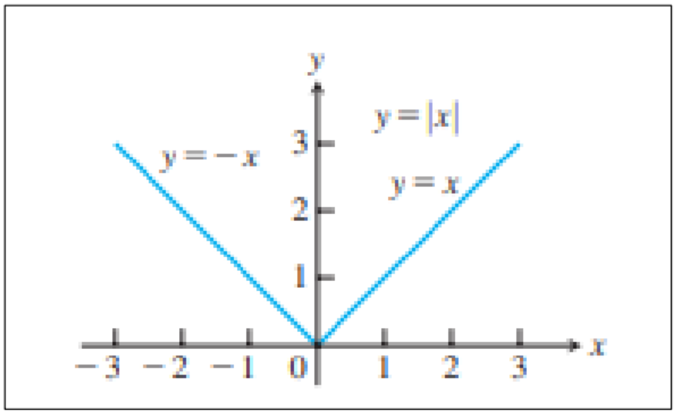


Figure 1-3 The absolute value function has domain (-∞, ∞) and range (0, ∞).

Example 3: The function

$$f(x) = \begin{cases} -x\,, x < 0 & \text{first formala} \\ x^2\,, 0 \leq x \leq 1 & \text{second formela} \\ 1, x > 1 & \text{third formela} \end{cases}$$

Plot this function

is defined on the entire real line but has values given by different formulas, depending on the position of x. The values of ƒ are given by y = -x when $x < 0$, y = x^2 when $0 \leq x \leq$, and y = 1 when $x > 1$. The function, however, is just one function whose domain is the entire set of real numbers.

x	$F(x)$
-4	4
-3	3
-2	2
-1	1
0	0
1	1
2	1
3	1
4	1

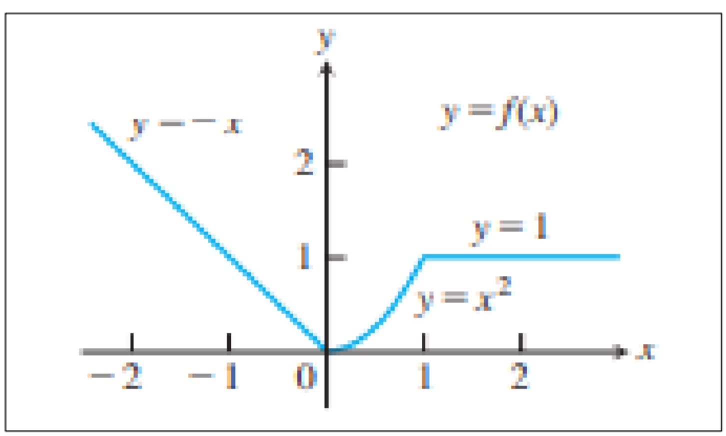


Figure 1-4 example 3

1.5 Increasing and Decreasing Functions

If the graph of a function climbs or rises as you move from left to right, we say that the function is increasing. If the graph descends or falls as you move from left to right, the function is decreasing.

Definition Let ƒ be a function defined on an interval I and let $x1$ and $x2$ be two distinct points in I:

1. If ƒ $(x2)$ > ƒ$(x1)$ whenever $x1 > x2$, then ƒ is said to be **increasing** on I.

2. If ƒ $(x2)$ < ƒ$(x1)$ whenever $x1 > x2$, then ƒ is said to be **decreasing** on I.

It is important to realize that the definitions of increasing and decreasing functions must be satisfied for every pair of points $x1$ and $x2$ in I with $x1 > x2$. Because we use the inequality < to compare the function values, instead of ≤, it is sometimes said that ƒ is strictly increasing or decreasing on I. The interval I may be finite (also called bounded) or infinite (unbounded).

The function graphed in Figure 1.4 (example 3) is **decreasing** on (-∞, 0) and **increasing** on (0, 1). The function is **neither increasing nor decreasing** on the interval (1, ∞) because the function is constant on that interval, and hence the strict inequalities in the definition of **increasing or decreasing are not satisfied** on (1, ∞).

1.6 Even Functions and Odd Functions:

Definition A function y = ƒ(x) is an

Even Function of x if ƒ$(-x)$ = ƒ(x),

Odd Function of x if ƒ$(-x)$ = −ƒ(x),

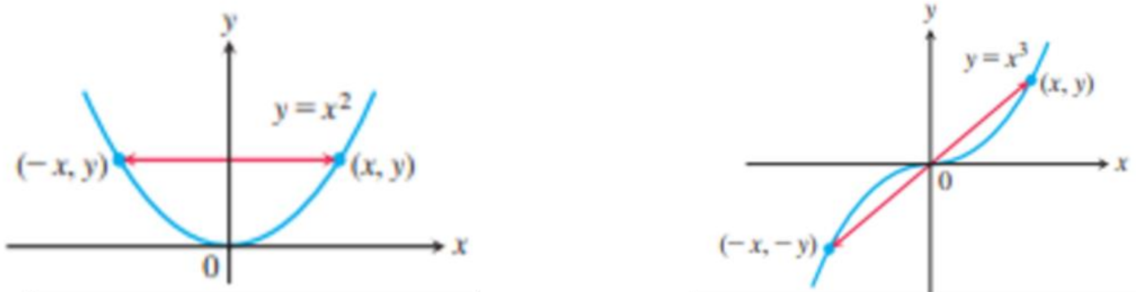
for every x in the function's domain.

The names even and odd come from the powers of x. If y is an even power of x, as in $y = x$ 2or y =, it is an **even function** of x because $(-x)^2 = (x)^2$ and $(-x)^4 = -x^4$.

If y is an odd power of x, as in $y = x$ or $y = x$ 3, it is an **odd function** of x because $(-x) = -x$ and $(-x)^3 = -x^3$.

The graph of an **even function** is **symmetric about the y-axis**. Since ƒ(-x) = ƒ(x), a point (x, y) lies on the graph if and only if the point (-x, y) lies on the graph (Figure 1.12a). A reflection across the y-axis leaves the graph unchanged.

The graph of an **odd function** is **symmetric about the origin**. Since ƒ(-x) = -ƒ(x), a point (x, y) lies on the graph if and only if the point (-x, -y) lies on the graph (Figure 1.12b). Equivalently, a graph is symmetric about the origin if a rotation of 180° about the origin leaves the graph unchanged. Notice that the definitions imply that both x and -x must be in the domain of ƒ.



Even function

odd function

Example 4: for the functions below, indicate whether the function is even or odd.

1. $f(x) = x^2$

2. $f(x) = x^2 + 1$

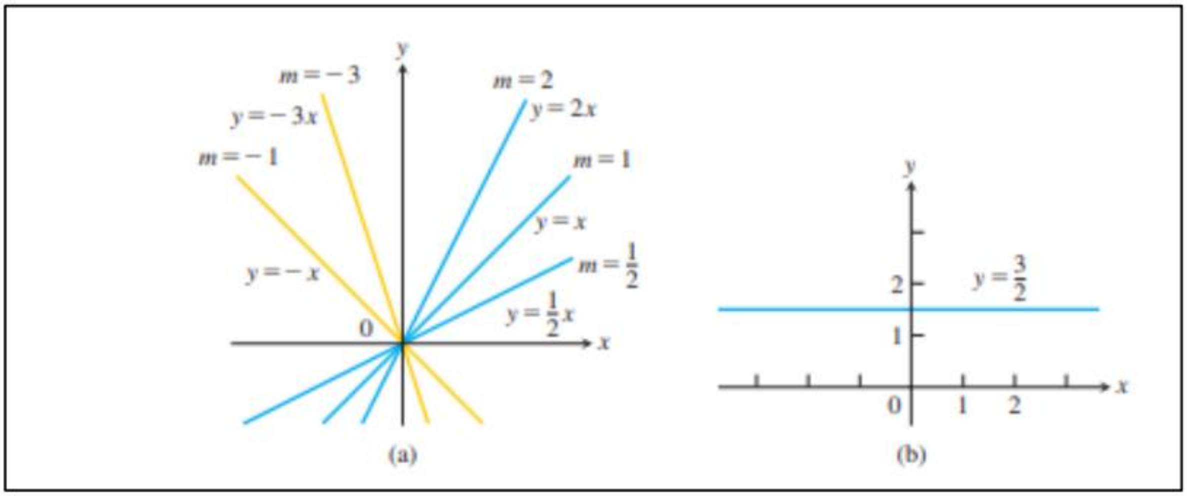
3. $f(x) = x$

4. $f(x) = x + 1$

1.7 Common Functions

1.7.1 Linear Functions

Linear Functions A function of the form $f(x) = mx + b$, where m and b are fixed constants, is called a linear function. The figure below shows an array of lines $f(x) = mx$. Each of these has b = 0, so these lines pass through the origin. The function $f(x) = x$ where $m = 1$ and $b = 0$ is called the **identity function**. **Constant functions** result when the slope is $m = 0$.



1.7.2 The Slope of a line

Increments – When a particle moves from one position in the plane to another, the net changes in the particle's coordinates are calculated by subtracting the coordinates of the starting point $(x1, y1)$ from the coordinates of the stopping point $(x2, y2)$,

$$\Delta x = x2 - x1, \Delta y = y2 - y1$$

Then the slope m is:

$$m = \frac{\Delta y}{\Delta x} = \frac{y2 - y1}{x2 - x1}$$

A line that goes uphill as x increases has a positive slope. A line that goes downhill as x increases has a negative slope.

- A horizontal line has slope zero because $y = 0$
- The slope of a vertical line is undefined because $x = 0$
- Parallel lines have same slope.
- If neither of two perpendicular lines L1 and L2 is vertical, their slopes $m1$ *and* $m2$ are related by the equation: $m1 \,.\, m2 = -1$

$\boldsymbol{m} = \mathbf{tan}\ \boldsymbol{\theta}$, where $\boldsymbol{\theta}$ is the angle of inculcation

- The angle of inclination of a horizontal line is taken to be 0°
- Parallel lines have equal angle of inclination.

Example 5: Find the slope of the line determined by two points A (2,1) and B (-1,3) and find the common slope of the line perpendicular to AB.

Example 6: Use slopes to determine in each case whether the points are collinear (lie on a common straight line):

- A(1,0) , B(0,1) , C(2,1) .
- A(-3,-2) , B(-2,0) , C(-1,2) , D(1,6) .