جامعة المستقبل
AL MUSTAQBAL UNIVERSITY
كلية العلوم

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Mathematics 1

Lecture 3

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الرياضيات الاساسية: المرحلة الاولى

مادة الرياضيات 1

المحاضرة الثالثة

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Contents

1 Functions

In mathematics, a function is a fundamental concept that describes a relationship between two sets.

These sets include:

1. **Natural Numbers ($\mathbb{N}$):** The set of all positive integers starting from 1.

$$\mathbb{N} = \{1, 2, 3, 4, \dots\}$$

2. **Whole Numbers ($\mathbb{W}$):** The set of all non-negative integers, including zero.

$$\mathbb{W} = \{0, 1, 2, 3, 4, \dots\}$$

3. **Integers ($\mathbb{Z}$):** The set of all whole numbers, including their negative counterparts.

$$\mathbb{Z} = \{\dots, -3, -2, -1, 0, 1, 2, 3, \dots\}$$

4. **Rational Numbers ($\mathbb{Q}$)**: The set of all numbers that can be expressed as a fraction $\frac{p}{q}$, where p and q are integers, and $q \neq 0$.

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\}$$

Examples: $\frac{1}{2}, -\frac{3}{4}, 5, 0$.

5. **Irrational Numbers:** The set of all numbers that cannot be expressed as a fraction. These numbers have non-repeating, non-terminating decimal expansions. Examples: $\sqrt{2}, \pi, e$.

6. **Real Numbers ($\mathbb{R}$):** The set of all rational and irrational numbers.

$$\mathbb{R} = \mathbb{Q} \cup \text{Irrational Numbers}$$

   Examples: $-2, 0, \frac{3}{4}, \sqrt{3}, \pi$.

7. **Complex Numbers ($\mathbb{C}$):** The set of all numbers in the form $a+bi$, where $a, b \in \mathbb{R}$ and $i = \sqrt{-1}$ (the imaginary unit). Formally:

$$\mathbb{C} = \{a + bi \mid a, b \in \mathbb{R}\}$$

   Examples: $2 + 3i, -1 - 4i, 5$ (since $5 = 5 + 0i$).

1.1 Intervals

An **interval** is a set of real numbers that contains all numbers between any two numbers in the set. Intervals can be classified as open, closed, half-open, or infinite, depending on whether their endpoints are included or excluded.

Types of Intervals

1. **Open Interval ((a, b)):** The set of all real numbers between a and b, excluding the endpoints a and b.

$$(a, b) = \{x \in \mathbb{R} \mid a < x < b\}$$

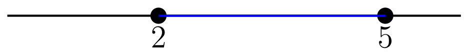
   Example: $(2, 5) = \{x \mid 2 < x < 5\}$.

2. **Closed Interval ($[a, b]$):** The set of all real numbers between a and b, including

the endpoints a and b.

$$[a, b] = \{x \in \mathbb{R} \mid a \leq x \leq b\}$$

Example: $[2, 5] = \{x \mid 2 \leq x \leq 5\}$.



3. **Half-Open Interval ($(a, b]$ or $[a, b)$)**: The set of all real numbers between a and b, including one endpoint and excluding the other.

$$(a, b] = \{x \in \mathbb{R} \mid a < x \leq b\}$$

Example: $(2, 5] = \{x \mid 2 < x \leq 5\}$.

$$[a, b) = \{x \in \mathbb{R} \mid a \leq x < b\}$$

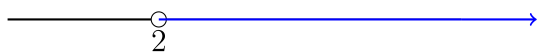
Example: $[2, 5) = \{x \mid 2 \leq x < 5\}$.

4. **Infinite Intervals**: Intervals that extend infinitely in one or both directions.

   - (a, ∞): All real numbers greater than a.

$$(a, \infty) = \{x \in \mathbb{R} \mid x > a\}$$

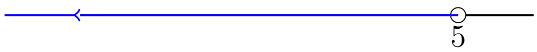
Example: $(2, \infty) = \{x \mid x > 2\}$.



- $(-\infty, b)$: All real numbers less than b.

$$(-\infty, b) = \{x \in \mathbb{R} \mid x < b\}$$

Example: $(-\infty, 5) = \{x \mid x < 5\}$.



- $[a, \infty)$: All real numbers greater than or equal to a.

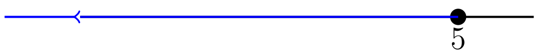
$$[a, \infty) = \{x \in \mathbb{R} \mid x \geq a\}$$

Example: $[2, \infty) = \{x \mid x \geq 2\}$.

- $(-\infty, b]$: All real numbers less than or equal to b.

$$(-\infty, b] = \{x \in \mathbb{R} \mid x \leq b\}$$

Example: $(-\infty, 5] = \{x \mid x \leq 5\}$.



1.2 Definition of Function

A **function** is a rule or correspondence that assigns to each element x in a set A (called the **domain**) exactly one element y in a set B (called the **codomain**).

We write:

$$f : A \to B \quad \text{where} \quad f(x) = y.$$

A is the **domain** of f, B is the **codomain** of f, The set of all outputs y is called the **range** of f.

1.3 Domain and Range of Functions

The **domain** and **range** of a function are fundamental concepts that describe the input and output of a function, respectively.

- **Domain**: The set of all possible input values (x) for which the function is defined.

$$\text{Domain of } f(x) = \{x \in \mathbb{R} \mid f(x) \text{ is defined}\}.$$

- **Range**: The set of all possible output values (y) that the function can produce.

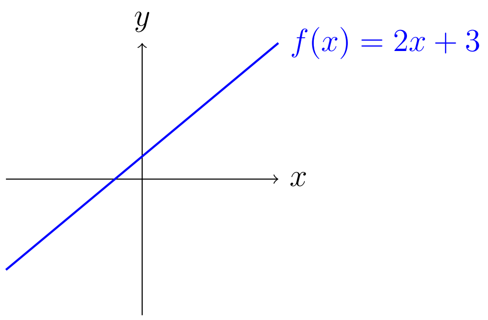
$$\text{Range of } f(x) = \{y \in \mathbb{R} \mid y = f(x) \text{ for some } x \in \text{Domain}\}.$$

Example 1.1 (Linear Function)**.**

$$f(x) = 2x + 3$$

Domain: $\mathbb{R}$ (all real numbers)

Range: $\mathbb{R}$ (all real numbers)

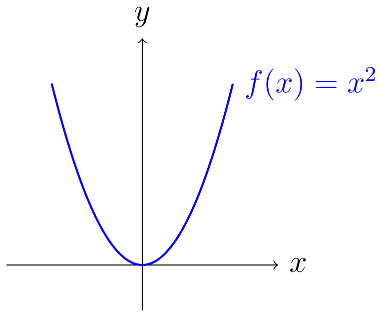


Example 1.2.

$$f(x) = x^2$$

Domain: $\mathbb{R}$ (all real numbers)

Range: $[0, \infty)$

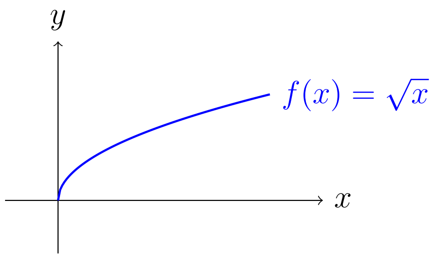


Example 1.3.

$$f(x) = \sqrt{x}$$

Domain: $[0, \infty)$

Range: $[0, \infty)$

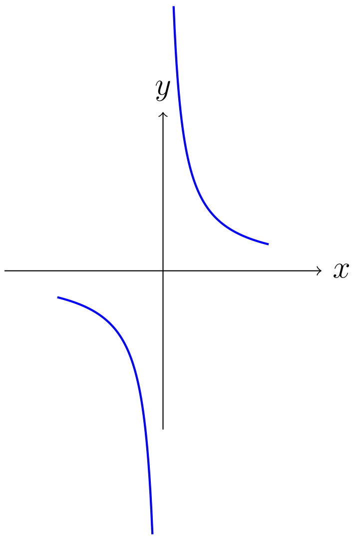


Example 1.4.

$$f(x) = \frac{1}{x}$$

Domain: $\mathbb{R} \setminus \{0\}$ (all real numbers except 0)

Range: $\mathbb{R} \setminus \{0\}$



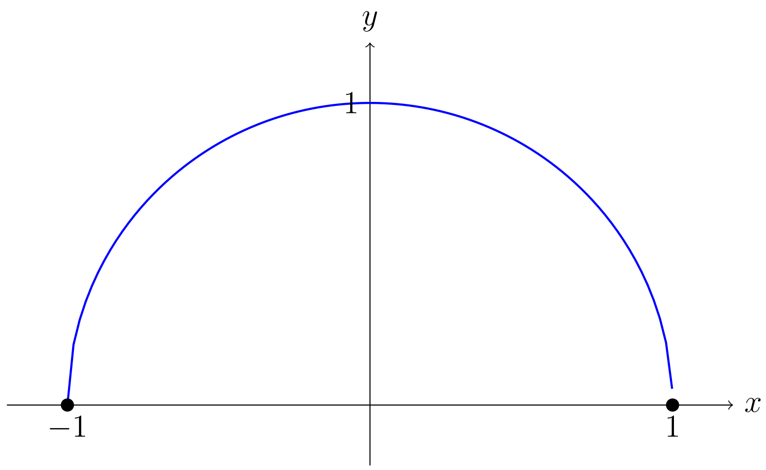
Example 1.5. The function $y = \sqrt{1 - x^2}$ represents the upper semicircle of a unit circle.

- **Domain**: $[-1, 1]$

  Since $1 - x^2 \geq 0$, the values of x are restricted to $-1 \leq x \leq 1$.

- **Range**: $[0, 1]$

  The square root ensures $y \geq 0$, and the maximum value of y occurs when $x = 0$, which gives $y = 1$.



Example 1.6. The function $y = \frac{1}{\sqrt{1-x^2}}$ is defined only where $1 - x^2 > 0$, which restricts the domain and avoids division by zero.

- **Domain**: $(-1, 1)$

  The expression $\sqrt{1 - x^2}$ is defined only for $-1 < x < 1$, excluding -1 and 1, where the denominator becomes zero.

- **Range**: $[1, \infty)$

  The function is strictly increasing on $(-1, 0)$ and strictly decreasing on $(0, 1)$, with the minimum value $y = 1$ at $x = 0$, and $y \to \infty$ as $x \to \pm 1$.

1.4 Properties and Operations of Functions

Addition of Functions The sum of two functions f and g is:

$$(f + g)(x) = f(x) + g(x)$$

Subtraction of Functions The difference of two functions f and g is:

$$(f-g)(x) = f(x) - g(x)$$

Multiplication of Functions The product of two functions f and g is:

$$(f \cdot g)(x) = f(x) \cdot g(x)$$

Division of Functions The division of two functions f and g (where $g(x) \neq 0$) is:

$$\left(\frac{f}{g}\right)(x) = \frac{f(x)}{g(x)}$$

Composition of Functions The composition of two functions f and g is:

$$(f \circ g)(x) = f(g(x))$$

For example: Let $f(x) = x^2$ and $g(x) = 3x$, then:

$$(f \circ g)(x) = f(g(x)) = f(3x) = (3x)^2 = 9x^2$$

Inverse of a Function The inverse of a function f, denoted f^{-1}, is defined as:

$$f(x) = y \quad \text{and} \quad f^{-1}(y) = x$$

For example: Let $f(x) = 2x + 3$, then the inverse function is:

$$f^{-1}(y) = \frac{y-3}{2}$$

For $y = 7$:

$$f^{-1}(7) = \frac{7-3}{2} = 2$$

Even and Odd Functions A function $f(x)$ is **even** if:

$$f(-x) = f(x) \quad \text{for all } x \in \text{Domain}(f)$$

A function $f(x)$ is **odd** if:

$$f(-x) = -f(x) \quad \text{for all } x \in \text{Domain}(f)$$

For examples:

$f(x) = x^2$ is even because:

$$f(-x) = (-x)^2 = x^2 = f(x)$$

$f(x) = x^3$ is odd because:

$$f(-x) = (-x)^3 = -x^3 = -f(x)$$

1.5 Trigonometric Functions

1. Sine Function: $\sin(x)$

$$\text{Domain of } \sin(x): \ (-\infty, \infty)$$

$$\text{Range of } \sin(x): \ [-1, 1]$$

2. Cosine Function: $\cos(x)$

$$\text{Domain of } \cos(x) : (-\infty, \infty)$$

$$\text{Range of } \cos(x) : [-1, 1]$$

3. Tangent Function: $\tan(x)$

$$\text{Domain of } \tan(x) : \Big((-\infty, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \infty)\Big)$$

$$\text{Range of } \tan(x) : (-\infty, \infty)$$

4. Cotangent Function: $\cot(x)$

$$\text{Domain of } \cot(x) : ((-\infty, \pi) \cup (\pi, \infty))$$

$$\text{Range of } \cot(x) : (-\infty, \infty)$$

5. Secant Function: $\sec(x)$

$$\text{Domain of } \sec(x) : (-\infty, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \infty)$$

$$\text{Range of } \sec(x) : (-\infty, -1] \cup [1, \infty)$$

6. Cosecant Function: $\csc(x)$

$$\text{Domain of } \csc(x) : (-\infty, \pi) \cup (\pi, \infty)$$

$$\text{Range of } \csc(x) : (-\infty, -1] \cup [1, \infty)$$

1.5.1 Trigonometric Identities

- $\sin^2(x) + \cos^2(x) = 1$
- $1 + \tan^2(x) = \sec^2(x)$
- $1 + \cot^2(x) = \csc^2(x)$

Reciprocal Identities:

- $\sec(x) = \frac{1}{\cos(x)}$
- $\csc(x) = \frac{1}{\sin(x)}$
- $\cot(x) = \frac{1}{\tan(x)}$

Quotient Identities:

- $\tan(x) = \frac{\sin(x)}{\cos(x)}$
- $\cot(x) = \frac{\cos(x)}{\sin(x)}$

Double Angle Identities

- $\sin(2x) = 2\sin(x)\cos(x)$
- $\cos(2x) = \cos^2(x) - \sin^2(x)$

Sum and Difference Identities

- $\sin(x \pm y) = \sin(x)\cos(y) \pm \cos(x)\sin(y)$
- $\cos(x \pm y) = \cos(x)\cos(y) \mp \sin(x)\sin(y)$

Product-to-Sum and Sum-to-Product Identities

- $\sin(x)\sin(y) = \frac{1}{2}[\cos(x-y) - \cos(x+y)]$
- $\cos(x)\cos(y) = \frac{1}{2}[\cos(x-y) + \cos(x+y)]$

- $\sin(x)\cos(y) = \frac{1}{2}\left[\sin(x+y) + \sin(x-y)\right]$

Angle (Degrees)	Angle (Radians)	$\sin(\theta)$	$\cos(\theta)$	$\tan(\theta)$
$0°$	0	0	1	0
$30°$	$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$45°$	$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1
$60°$	$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$
$90°$	$\frac{\pi}{2}$	1	0	undefined
$120°$	$\frac{2\pi}{3}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$-\sqrt{3}$
$135°$	$\frac{3\pi}{4}$	$\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	-1
$150°$	$\frac{5\pi}{6}$	$\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
$180°$	π	0	-1	0
$210°$	$\frac{7\pi}{6}$	$-\frac{1}{2}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{\sqrt{3}}$
$225°$	$\frac{5\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$-\frac{\sqrt{2}}{2}$	1
$240°$	$\frac{4\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$-\frac{1}{2}$	$\sqrt{3}$
$270°$	$\frac{3\pi}{2}$	-1	0	undefined
$300°$	$\frac{5\pi}{3}$	$-\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$-\sqrt{3}$
$315°$	$\frac{7\pi}{4}$	$-\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	-1
$330°$	$\frac{11\pi}{6}$	$-\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$-\frac{1}{\sqrt{3}}$
$360°$	2π	0	1	0

1.6 Absolute Value

The absolute value of a number represents its distance from zero on the number line, without considering its sign. Mathematically, the absolute value of a real number x is defined as:

$$|x| = \begin{cases} x & \text{if } x \geq 0, \\ -x & \text{if } x < 0. \end{cases}$$

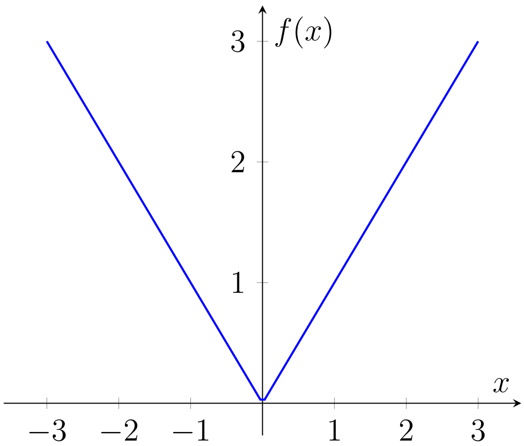
This function transforms all negative numbers into positive ones while leaving positive numbers unchanged.

$$\text{Domain of } f(x) = |x| : \ (-\infty, \infty)$$

Range: The range of the absolute value function is all non-negative real numbers:

$$\text{Range of } f(x) = |x| : \ [0, \infty)$$

Graph of the Absolute Value Function



Exercises

Intervals

1. For each of the following intervals, state whether it is open, closed, half-open, or half-closed.

   (a) $(-5, 3)$

   (b) $[1, 7]$

   (c) $(-\infty, 4)$

   (d) $(-2, 6]$

   (e) $[0, \infty)$

(f) $(-1, 2) \cup (3, 5)$

2. For each of the following intervals, sketch its graph on the real number line.

   (a) $(-3, 2)$

   (b) $[-1, 4)$

   (c) $(-\infty, 0)$

   (d) $[2, \infty)$

Domain and Range of Composite Functions

For the following functions, find the domain and range:

1. $f(x) = \sqrt{x^2 + 4x + 4}$

2. $g(x) = \frac{1}{x+3}$

3. $h(x) = \sin(x)$

4. $p(x) = \ln(x^2 + 1)$

5. $q(x) = \frac{1}{x-1}$

Composition of Functions

For the following functions, compute the compositions $f \circ g$ and $g \circ f$:

1. $f(x) = 2x + 3$, $g(x) = x^2 - 1$

2. $f(x) = \sqrt{x}$, $g(x) = 3x + 4$

Even and Odd Functions

For each of the following functions, determine whether the function is even, odd, or neither:

1. $f(x) = x^2 + 3$

2. $g(x) = x^3 - 2x$

3. $h(x) = \frac{1}{x}$

4. $p(x) = \cos(x)$

5. $q(x) = \sin(x)$

6. $r(x) = x^4 - x^2 + 1$

7. $s(x) = x^3 + 2x$

Graphing of Functions

Sketch the graph of the following functions:

1. $f(x) = 2x + 3$

2. $g(x) = -x^2 + 6x - 8$

3. $h(x) = 2|x + 1|$

4. $p(x) = -|x| + 4$

5. $$h(x) = \begin{cases} x + 2 & \text{if } x < 1 \\ 4 - x & \text{if } x \geq 1 \end{cases}$$

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