Al-Mustaqbal University

College of Engineering & Technology

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Lecture Title: Real Number, Intervals, Solving Inequalities, and Absolute Value

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Chapter one

1- Real Number (R)

a real number is a number that can be used to measure a continuous one-dimensional quantity such as a distance, duration or temperature.

i . Natural Numbers (N) such that:

$$\mathrm{N} = \{1, 2, 3, 4, \ldots\}$$

ii. Intager Numbers (I or Z) such that:

$$\mathrm{I \text{ or } Z} = \{\ldots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \ldots\}$$

iii. Rational Numbers (Q): it is all numbers of the form

$\frac{p}{q}$, such that p and q are integers and $q \neq 0$:

$$\mathbb{Q} = \{x \in \mathbb{R} : x = \frac{p}{q}, \text{ where } p, q \in \mathbb{Z} \text{ and } q \neq 0\}$$

Example: $\frac{1}{2}, \frac{5}{3}, 0, \frac{50}{10}, \quad \ldots$

__Note__: The rational Numbers can be written as decimal from

$(\frac{1}{3} = 0.333, \frac{1}{4} = 0.25, \ldots)$.

iv. Irrational Numbers (Q'): A number which is not

rational is said to be irrational.

Example: $\{\sqrt{2}, \sqrt{3}, \sqrt{5}, \sqrt{7}, \pi = 3.14, \ldots\}$

__Note__: $\emptyset \subset \mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ and $\mathbb{Q} \mathbb{U} \mathbb{Q}' = \mathbb{R}$

Properties of Real Numbers with Addition: ***(R, +)***

Let a, b, c ∈ R, then:

1. $a + b \in \text{R}$ (Closure)
2. $a + b = b + a$ (Commutative)
3. $a + (b + c) = (a + b) + c$ (Associative)
4. $a + 0 = 0 + a = 0$ (Identity Element)
5. $\exists(-a) \in \mathbb{R}$ such that $a+(-a) = (-a)+a = 0$ (Additive Inverse)

Properties of Real Numbers with Multiplication: ***(R, .)***

Let a, b, c ∈ ℝ, then:

1. $a.b \in \mathbb{R}$ (Closure)
2. $a.b = b.a$ (Commutative)
3. $a.(b.c) = (a.b).c$ (Associative)
4. $1.a = a.1$ (Multiplicative Identity)
5. $a.(b + c) = a.b + a.c$ (Distributive)

   $(b + c).a = b.a + c.a$
6. $\exists a^{-1} \in \mathbb{R}$ such that $a.a^{-1} = a.\frac{1}{a} = 1$ (Multiplication Inverse)

Intervals:-

1. **Finite intervals:-** Let $a, b \in \mathbb{R}$ such that $a < b$ then:

   (a) **Open Interval** $= \{x \in \mathbb{R} : a < x < b\} = (a, b)$

   (Note: $a \notin (a, b)$ and $b \notin (a, b)$)

   (b) **Closed Interval** $= \{x \in \mathbb{R} : a \leq x \leq b\} = [a, b]$

   (Note: $a \in [a, b]$ and $b \in [a, b]$)

   (c) **The Half Open Interval** $= \{x \in \mathbb{R} : a < x \leq b\} = (a, b]$

   (Note: $b \in (a, b]$ and $a \notin (a, b]$)

   OR:

   The Half Open Interval $= \{x \in \mathbb{R} : a \leq x < b\} = [a, b)$

   (Note: $b \notin [a, b)$ and $a \in [a, b)$)

2. **Infinite intervals:-** Let each of $a, b \in \mathbb{R}$ such that $a < b$ then:

   (a) $\{x \in \mathbb{R}$ such that $a < x < \infty$ (or $x > a$) $\} = (a, \infty)$

   (b) $\{x \in \mathbb{R}$ such that $a \leq x < \infty$ (or $x \geq a$) $\} = [a, \infty)$

   (c) $\{x \in \mathbb{R}$ such that $-\infty < x < a$ (or $x < a$) $\} = (-\infty, a)$

   (d) $\{x \in \mathbb{R}$ such that $-\infty < x \leq a$ (or $x \leq a$) $\} = (-\infty, a]$

   (e) $\{x \in \mathbb{R}$ such that $-\infty < x < \infty\} = (-\infty, \infty) = \mathbb{R}$

Inequalities:-

Let $a, b \in \mathbb{R}$, b is **greater** than a and denoted by $b > a$ **if** $b - a > 0$.

Solving Inequalities:-

Solving the inequalities means obtaining all values of x for which the inequality is true.

Properties of Inequalities:-

Let $a, b, c \in \mathbb{R}$, then:

1. if $a < b$, then $a + c < b + c$
2. if $a < b$ and $c > 0$, then $a.c < b.c$
3. if $a < b$ and $c < 0$, then $a.c > b.c$

Note :- In general, we have linear and non-linear inequalities.

Linear Inequalities Examples:-

Example 1: Solve the following inequality: $3(x+2) < 5$?

solution:-

$3(x+2) < 5 \longrightarrow 3(x+2) < 5 \longrightarrow 3x < 5 - 6 \longrightarrow < \frac{-1}{3}$

Hence, the solution set $=\{x \in \mathbb{R} : x < \frac{-1}{3}\} = (-\infty, \frac{-1}{3})$.

Example 2: Solve the following inequality: $7 < 2x + 3 < 11$?

solution:-

$7 < 2x + 3 < 11 \longrightarrow -3 + 7 < 2x < -3 + 11 \longrightarrow 4 < 2x < 8 \longrightarrow 2 < x < 4$

Hence, the solution set $=\{x \in \mathbb{R} : 2 < x < 4\} = (2, 4)$.

Non-Linear Inequalities Examples:-

Example 1: Solve the following inequality: $x^2 < 25$?

solution:- $x^2 < 25 \to x^2 - 25 < 0 \to (x-5)(x+5) < 0$

Since the result is negative, then there are two possibilities:

Either:

$(x+5) > 0$ and $(x-5) < 0 \longrightarrow x > -5$ and $x < 5$

So, the solution set is $(-5, 5)$

Or:

$(x+5), 0$ and $(x-5) > 0 \longrightarrow x < -5$ and $x > 5$

So, the solution set is $\emptyset$

Therefore, the solution set for the inequality is

$(-5, 5) \cup \emptyset = (-5, 5)$

Example 2: Solve the following inequality: $x^2 - 5x > 6$?

solution:-

$x^2 - 5x > 6 \to x^2 - 5x - 6 > 0 \to (x-6)(x+1) > 0$

Since the result is Positive, then there are two possibilities:

Either:

$(x-6) > 0$ and $(x+1) > 0 \longrightarrow x > 6$ and $x > -1$

So, the solution set: $S_1 = \{x \in \mathbb{R} : x > 6\} = (6, \infty)$

Or:

$(x-6)<0$ and $(x+1),0 \longrightarrow x<6$ and $x<-1$

So, the solution set: $S_2=\{x\in\mathbb{R}: x<-1\}=(-\infty,-1)$

Therefore, the solution set for the inequality is:

$S=S_1\cup S_2=(6,\infty)\cup(-\infty,-1)=\mathbb{R}\setminus[-1,6]$

Absolute Value:-

The absolute value of a real number x is denoted by $|x|$ and defined as follows:

$$|x|=\sqrt{x^2}=\begin{cases} x & \text{if } x>0 \\ 0 & \text{if } x=0 \\ -x & \text{if } x<0 \end{cases}$$

Examples: $|-8|=8$, $|\frac{-2}{3}|=\frac{2}{3}$, $|9|=9$, $|0|=0$, etc.

Properties of Absolute Value:-

1. $|-a|=|a|$

   proof: $|-a|=\sqrt{(-a)^2}=\sqrt{a^2}=|a|$

2. $||a||=|a|$

   proof: $||a||=\sqrt{|a|^2}=\sqrt{a^2}=|a|$

3. $|a.b|=|a|.|b|$

   proof: $|a.b|=\sqrt{(a.b)^2}=\sqrt{a^2.b^2}=\sqrt{a^2}.\sqrt{b^2}=|a|.|b|$

4. $|\frac{a}{b}|=\frac{|a|}{|b|}$; $b\neq 0$

   proof: $|\frac{a}{b}|=\sqrt{(\frac{a}{b})^2}=\sqrt{\frac{a^2}{b^2}}=\frac{\sqrt{a^2}}{\sqrt{b^2}}=\frac{|a|}{|b|}$

5. $|a+b|\leq|a|+|b|$

Solving Absolute Value Inequalities:-

The absolute value of x can be written as follows:

$$|x|=\sqrt{x^2}=\begin{cases} x & \text{if } x\geq 0 \\ -x & \text{if } x<0 \end{cases}$$

The above definition means the absolute value of any real number is a real non-negative number.

Geometrically, the absolute value of unmber x is the distance point between "x" and the origin point "0". In general, $|a-b|$ is the distance between a and b on the real number line "$\mathbb{R}$".

Remarks:

1. To solve the inequality $|x|<a$ where $a,x\in\mathbb{R}$.

   Case (1): If $x\geq 0\Longrightarrow|x|=x$,

   but $|x|<a\Longrightarrow x<a.\Longrightarrow S_1=(-\infty,a)$

   Case (2): If $x<0\Longrightarrow|x|=-x$, but $|x| < a \Longrightarrow -x < a \Longrightarrow x > -a. \Longrightarrow S_2 = (-a, \infty)$

Since, $S = S_1 \cap S_2$

$\Longrightarrow \{x \in \mathbb{R} : |x| < a\} = \{x \in \mathbb{R} : -a < x < a\} = (-a, a)$

Similarly,

$\Longrightarrow \{x \in \mathbb{R} : |x| \leq a\} = \{x \in \mathbb{R} : -a \leq x \leq a\} = [-a, a]$

2. To solve the inequality $|x| > a$ where $a, x \in \mathbb{R}$.

   Case (1): If $x \geq 0 \Longrightarrow |x| = x$,

   but $|x| > a \Longrightarrow x > a. \Longrightarrow S_1 = (a, \infty)$

   Case (2): If $x < 0 \Longrightarrow |x| = -x$,

   but $|x| > a \Longrightarrow -x > a \Longrightarrow x < -a. \Longrightarrow S_2 = (-\infty, -a)$

   Since, $S = S_1 \cup S_2$

   $\Longrightarrow \{x \in \mathbb{R} : |x| > a\} = (a, \infty) \cup (-\infty, -a) = \mathbb{R} \setminus [-a, a]$

   Similarly,

   $\Longrightarrow \{x \in \mathbb{R} : |x| \geq a\} = [a, \infty) \cup (-\infty, -a] = \mathbb{R} \setminus (-a, a)$

Examples:- Find the solution set for the following inequalities?

- $|x| > 3$

  solution:-

  $\{x \in \mathbb{R} : |x| > 3\} = \{x \in \mathbb{R} : x > 3 \text{ or } x < -3\} =$

  $(3, \infty) \cup (-\infty, -3) = \mathbb{R} \setminus [-3, 3]$

- $|x| \leq 4$

  solution:-

  $\{x \in \mathbb{R} : |x| \leq 4\} = \{x \in \mathbb{R} : -4 \leq x \leq 4\} = [-4, 4]$

- $|x - 4| < 5$

  solution:-

  $\{x \in \mathbb{R} : |x - 4| < 5\} = \{x \in \mathbb{R} : -5 < x - 4 < 5\}$

  $= \{x \in \mathbb{R} : -1 < x < 9\} = (-1, 9)$

- $|7 - 4x| \geq 1$

  solution:-

  $\{x \in \mathbb{R} : |x - 4| \geq 1\} = \{x \in \mathbb{R} : 7 - 4x \geq 1 \text{ or } 7 - 4x \leq -1\}$

  $= \{x \in \mathbb{R} : -4x \geq -6 \text{ or } -4x \leq -8\}$

  $= \{x \in \mathbb{R} : x \leq \frac{3}{2} \text{ or } x \geq 2\}$

  $= (-\infty, \frac{3}{2}] \cup [2, \infty)$

  $= \mathbb{R} \setminus (\frac{3}{2}, 2)$

Problems 1.1:

1. Write the following sets equivalent interval, and test of these intervals whether they are Open, Close or Half Open Intervals:

   (a) $\{x : -20 \leq x \leq -12\}$

   (b) $\{x : -3 \leq x < 4\}$

   (c) $\{x : -1 < x < 10\}$

   (d) $\{x : -2 < x \leq 0\}$

2. Give a description of the following intervals as sets:

(a) $(3, 5)$
(b) $(-3, 0)$
(c) $[2, 7]$
(d) $[-5, -1)$
(e) $(-4, 4)$
(f) $(-0, 7]$

3. Find the solution set of the following inequalities:

(a) $x(x-3) > 4$
(b) $2 < \frac{1}{x};\ x \neq 0$
(c) $x^2 \geq 25$
(d) $x^2 - 2x - 24 < 0$
(e) $-7 \leq -3x + 5 \leq 14$
(f) $\frac{x}{x-3} < 4$
(g) $\frac{x^2+2x-35}{x+2} > 0$
(h) $6x - 4 > 7x + 2$
(i) $x^2 \leq 16$
(j) $3x^2 > 2x + 5$
(k) $x^2 > 5x + 6$
(l) $\frac{x-3}{x+2} < 5$
(m) $\frac{1}{x-2} > \frac{2}{x+3}$
(n) $\frac{x-2}{x+3} < \frac{1}{2}$

4. Find the solution set of the following inequalities:

(a) $|x| \geq 5$
(b) $|x| < 2$
(c) $|3x+3| \geq 2$
(d) $1 \leq |\frac{x-3}{1-2x}| \leq 2$
(e) $|\frac{2-x}{x-3}| \geq 4$
(f) $|x+1| < |3x+4|$
(g) $\frac{|2-x|}{3x} \leq 1$
(h) $|\frac{3+2x}{3x}| \leq 1$
(i) $|x-1| \geq 6$
(j) $|2-2x| \leq 7$
(k) $|\frac{4}{2x+1}| \leq 3$