Limits and continuity :

Limits *:* The limit of $F(t)$ as t approaches C is the number L if :

Given any radius $\varepsilon > 0$ about L there exists a radius $\delta > 0$ about C such that for all t, $0 < |t - C| < \delta$ implies $|F(t) - L| < \varepsilon$ and we can write it as :

$$\lim_{t \to C} F(t) = L$$

The limit of a function $F(t)$ as $t \to C$ never depend on what happens when $t = C$.

Right hand limit : $\lim_{t \to C^+} F(t) = L$

The limit of the function $F(t)$ as $t \to C$ from the right equals L if : Given any $\varepsilon > 0$ (radius about L) there exists a $\delta > 0$ (radius to the right of C) such that for all t :

$$C < t < C + \delta \Rightarrow \quad |F(t) - L| < \varepsilon$$

Left hand limit : $\lim_{t \to C^-} F(t) = L$

The limit of the function $F(t)$ as $t \to C$ from the left equal L if :
Given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all t :

$$C - \delta < t < C \Rightarrow \quad |F(t) - L| < \varepsilon$$

The limit combinations theorems :

1) $\lim [F_1(t) \mp F_2(t)] = \lim F_1(t) \mp \lim F_2(t)$

2) $\lim [F_1(t) * F_2(t)] = \lim F_1(t) * \lim F_2(t)$

3) $\lim \dfrac{F_1(t)}{F_2(t)} = \dfrac{\lim F_1(t)}{\lim F_2(t)} \quad where \quad \lim F_2(t) \neq 0$

4) $\lim [k * F_1(t)] = k * \lim F_1(t) \qquad \forall k$

5) $\lim_{\theta \to 0} \dfrac{Sin\,\theta}{\theta} = 1$

provided that θ is measured in radius

The limits (in $1 - 4$) are all to be taken as $t \to C$ and $F_1(t)$ and $F_2(t)$ are to be real functions .

Email: firas.thair.almaliky@uomus.edu.iq

Thm. -1 : The sandwich theorem **:** Suppose that $f(t) \le g(t) \le h(t)$ for all $t \ne C$ in some interval about C and that $f(t)$ and $h(t)$ approaches the same limit L as $t \to C$, then :

$$\lim_{t \to C} g(t) = L$$

Infinity as a limit **:**

1. The limit of the function $f(x)$ as x approaches infinity is the number L: $\lim_{x \to \infty} f(x) = L$. If , given any $\varepsilon > 0$ there exists a number M such that for all x : $M < x \quad \Rightarrow \quad |f(x) - L| < \varepsilon$.
2. The limit of $f(x)$ as x approaches negative infinity is the number L : $\lim_{x \to -\infty} f(x) = L$. If , given any $\varepsilon > 0$ there exists a number N such that for all x : $x < N \quad \Rightarrow \quad |f(x) - L| < \varepsilon$.

The following facts are some times abbreviated by saying :

a) As x approaches 0 from the right , $1/x$ tends to ∞ .
b) As x approaches 0 from the left , $1/x$ tends to $-\infty$.
c) As x tends to ∞ , $1/x$ approaches 0 .
d) As x tends to $-\infty$, $1/x$ approaches 0 .

Continuity **:**

Continuity at an interior point **:** A function $y = f(x)$ is continuous at an interior point C of its domain if : $\lim_{x \to C} f(x) = f(C)$.

Continuity at an endpoint **:** A function $y = f(x)$ is continuous at a left endpoint a of its domain if : $\lim_{x \to a^+} f(x) = f(a)$.

A function $y = f(x)$ is continuous at a right endpoint b of its domain if: $\lim_{t \to b^-} f(x) = f(b)$.

Continuous function **:** A function is continuous if it is continuous at each point of its domain .

Discontinuity at a point **:** If a function f is not continuous at a point C , we say that f is discontinuous at C , and call C a point of discontinuity of f .

The continuity test **: The function $y = f(x)$ is continuous at $x = C$ if and only if all three of the following statements are true :**

1) $f(C)$ **exist (C is in the domain of f).**
2) $\lim\limits_{x \to C} f(x)$ **exists (f has a limit as $x \to C$).**
3) $\lim\limits_{x \to C} f(x) = f(C)$ **(the limit equals the function value).**

Thm.-2 ***: The limit combination theorem for continuous function :***

If the function f and g are continuous at $x = C$, then all of the following combinations are continuous at $x = C$:

$$1)\ f \mp g \quad 2)\ f.g \quad 3)\ k.g \ \ \forall k \quad 4)\ g_o f\ ,\ f_o g \quad 5)\ f/g$$

$$provided \quad g(C) \neq 0$$

Thm.-3 ***:*** **A function is continuous at every point at which it has a derivative . That is , if $y = f(x)$ has a derivative $f'(C)$ at $x = C$, then f is continuous at $x = C$.**

Example: Find

$$1)\ \lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2} \quad , \quad 2)\ \lim_{x\to a}\frac{x^3-a^3}{x^4-a^4}$$

$$3)\ \lim_{x\to 0}\frac{Sin5x}{Sin3x} \quad , \quad 4)\ \lim_{y\to 0}\frac{tan2y}{3y}$$

$$5)\ \lim_{x\to 0}\frac{Sin2x}{2x^2+x} \quad , \quad 6)\ \lim_{x\to\infty}\left(1+Cos\frac{1}{x}\right)$$

$$7)\ \lim_{x\to\infty}\frac{3x^3+5x^2-7}{10x^3-11x+5} \quad , \quad 8)\ \lim_{y\to\infty}\frac{3y+7}{y^2-2}$$

$$9)\ \lim_{x\to\infty}\frac{x^3-1}{2x^2-7x+5} \quad , \quad 10)\ \lim_{x\to -1^-}\frac{1}{x+1}$$

$$11)\ \lim_{x\to 0}Cos\left(1-\frac{Sinx}{x}\right) \quad , \quad 12)\ \lim_{x\to 0}Sin\left(\frac{\pi}{2}Cos(tan\,x)\right)$$

Email: firas.thair.almaliky@uomus.edu.iq

S0l.-

1) $$\lim_{x\to 0}\frac{5x^3+8x^2}{3x^4-16x^2}=\lim_{x\to 0}\frac{5x+8}{3x^2-16}=\frac{0+8}{0-16}=-\frac{1}{2}$$

2) $$\lim_{x\to a}\frac{x^3-a^3}{x^4-a^4}=\lim_{x\to a}\frac{(x-a)(x^2+ax+a^2)}{(x-a)(x+a)(x^2+a^2)}=\frac{a^2+a^2+a^2}{(a+a)(a^2+a^2)}=\frac{3}{4a}$$

3) $$\lim_{x\to 0}\frac{5\,\dfrac{Sin5x}{5x}}{3\,\dfrac{Sin3x}{3x}}=\frac{5}{3}\cdot\frac{\lim\limits_{5x\to 0}\dfrac{Sin5x}{5x}}{\lim\limits_{3x\to 0}\dfrac{Sin3x}{3x}}=\frac{5}{3}$$

4) $$\lim_{y\to 0}\frac{\tan 2y}{3y}=\frac{2}{3}\cdot\lim_{2y\to 0}\frac{Sin2y}{2y}\cdot\lim_{y\to 0}\frac{1}{Cos2y}=\frac{2}{3}$$

5) $$\lim_{x\to 0}\frac{Sin2x}{2x^2+x}=2\lim_{2x\to 0}\frac{Sin2x}{2x}\cdot\lim_{x\to 0}\frac{1}{2x+1}=2$$

6) $$\lim_{x\to\infty}\left(1+Cos\frac{1}{x}\right)=1+Cos0=2$$

7) $$\lim_{x\to\infty}\frac{3x^3+5x^2-7}{10x^3-11x+5}=\lim_{x\to\infty}\frac{3+\dfrac{5}{x}-\dfrac{7}{x^3}}{10-\dfrac{11}{x^2}+\dfrac{5}{x^3}}=\frac{3}{10}$$

8) $$\lim_{y\to\infty}\frac{3y+7}{y^2-2}=\lim_{y\to\infty}\frac{\dfrac{3}{y}+\dfrac{7}{y^2}}{1-\dfrac{2}{y^2}}=\frac{0}{1}=0$$

9) $$\lim_{x\to\infty}\frac{x^3-1}{2x^2-7x+5}=\lim_{x\to\infty}\frac{1-\dfrac{1}{x^3}}{\dfrac{2}{x}-\dfrac{7}{x^2}+\dfrac{5}{x^3}}=\frac{1}{0}=\infty$$

Email: firas.thair.almaliky@uomus.edu.iq

10) $\lim\limits_{x\to -1^-} \dfrac{1}{x+1} = \dfrac{1}{-1+1} = -\infty$

11) $\lim\limits_{x\to 0} Cos\left(1-\dfrac{Sinx}{x}\right) = Cos\left(1-\lim\limits_{x\to 0}\dfrac{Sinx}{x}\right) = Cos0 = 1$

12) $\lim\limits_{x\to 0} Sin\left(\dfrac{\pi}{2}Cos(\tan x)\right) = Sin\left(\dfrac{\pi}{2}Cos(\tan 0)\right) = Sin\left(\dfrac{\pi}{2}Cos0\right) = Sin\dfrac{\pi}{2} = 1$

<u>*EX-*</u> Test continuity for the following function :

$$f(x)=\begin{cases} x^2-1 & -1\le x<0 \\ 2x & 0\le x<1 \\ 1 & x=1 \\ -2x+4 & 1<x\le 2 \\ 0 & 2<x\le 3 \end{cases}$$

<u>*Sol.*</u>- We test the continuity at midpoints $x = 0, 1, 2$ and endpoints $x = -1, 3$.

At $x=0 \Rightarrow$ $f(0)=2*0=0$

$\lim\limits_{x\to 0^-} f(x) = \lim\limits_{x\to 0}(x^2-1) = -1$

$\lim\limits_{x\to 0^+} f(x) = \lim\limits_{x\to 0} 2x = 0 \ne \lim\limits_{x\to 0^-} f(x)$

Since $\lim\limits_{x\to 0} f(x)$ doesn't exist

Hence the function discontinuous at $x=0$

At $x=1 \Rightarrow$ $f(1)=1$

$\lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1} 2x = 2$

$\lim\limits_{x\to 1^+} f(x) = \lim\limits_{x\to 1}(-2x+4) = 2 = \lim\limits_{x\to 1^-} f(x) = \lim\limits_{x\to 1} f(x)$

Since $\lim\limits_{x\to 1} f(x) \ne f(1)$

Hence the function is discontinuous at $x=1$

At $x=2 \Rightarrow$ $f(2)=-2*2+4=0$

$\lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2}(-2x+4) = 0$

$\lim\limits_{x\to 2^+} f(x) = \lim\limits_{x\to 2} 0 = 0 = \lim\limits_{x\to 2^-} f(x) = \lim\limits_{x\to 2} f(x)$

Since $\lim\limits_{x\to 2} f(x) = f(2) = 0$

Hence the function is continuous at $x=2$

Email: firas.thair.almaliky@uomus.edu.iq

$$At \quad x=-1 \Rightarrow \quad f(-1)=(-1)^2-1=0$$

$$\lim_{x\to -1^+} f(x)=\lim_{x\to -1}(x^2-1)=0=f(-1)$$

Hence the function is continuous at $x=-1$

$$At \quad x=3 \Rightarrow \quad f(3)=0$$

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3} 0=0=f(3)$$

Hence the function is continuous at $x=3$

EX- What value should be assigned to a to make the function :

$$f(x)=\begin{cases} x^2-1 & x<3 \\ 2ax & x\geq 3 \end{cases} \quad \text{continuous at } x=3\ ?$$

Sol. –

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x) \Rightarrow \lim_{x\to 3}(x^2-1)=\lim_{x\to 3} 2ax \Rightarrow \quad 8=6a \Rightarrow \quad a=\frac{4}{3}$$

H.W

1. Discuss the continuity of :

$$f(x)=\begin{cases} x+\dfrac{1}{x} & for \quad x<0 \\ -x^3 & for \quad 0\leq x<1 \\ -1 & for \quad 1\leq x<2 \\ 1 & for \quad x=2 \\ 0 & for \quad x>2 \end{cases}$$

(ans.: discontinuous at x=0,2 ; continuous at x=1)

Email: firas.thair.almaliky@uomus.edu.iq

2. Evaluate the following limits :

a) $\lim\limits_{x\to\infty} \dfrac{x+Sinx}{2x+5}$ b) $\lim\limits_{x\to\infty} \dfrac{1+Sinx}{x}$

c) $\lim\limits_{x\to 0} \dfrac{x}{\tan 3x}$ d) $\lim\limits_{x\to\infty} \dfrac{x.Sinx}{(x+Sinx)^2}$

e) $\lim\limits_{x\to 1} \dfrac{1-\sqrt{x}}{1-x}$ f) $\lim\limits_{x\to 1} \dfrac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

g) $\lim\limits_{n\to\infty} (\sqrt{n^2+1}-n)$

(ans.:a)1/2, b)0, c)1/3, d)0, e)1/2, f)-1/2√2, g)0)

Reference

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