



Subject Name: Calculus I1

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Lecturer: Dr. Amir N.Saud

Email: amir-najah@uomus.edu.iq

Lecture No. 4

Lecture Title: Limits of Functions

Limits of functions

Limit of a Function:.

If f is a function, then we say:

A is the limit of $f(x)$ as x approaches a if the value of $f(x)$ gets arbitrarily close to A as x approaches a . This is written in mathematical notation as:

$$\lim_{x \rightarrow a} f(x) = A$$

For example:

$$\lim_{x \rightarrow 3} x^2 = 9$$

$$\lim_{x \rightarrow 0} x = 0$$

$$\lim_{x \rightarrow 2} 4 = 4$$

The limit law:.

If L, M, c and k are real numbers and

$$\lim_{x \rightarrow c} f(x) = L \quad \text{and} \quad \lim_{x \rightarrow c} g(x) = M, \quad \text{then}$$

1. *Sum Rule:*
$$\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$$

The limit of the sum of two functions is the sum of their limits.

2. *Difference Rule:*
$$\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$$

The limit of the difference of two functions is the difference of their limits.

3. *Product Rule:*
$$\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$$

The limit of a product of two functions is the product of their limits.

4. *Constant Multiple Rule:*
$$\lim_{x \rightarrow c} (k \cdot f(x)) = k \cdot L$$

The limit of a constant times a function is the constant times the limit of the function.

5. *Quotient Rule:*
$$\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M}, \quad M \neq 0$$

The limit of a quotient of two functions is the quotient of their limits, provided the limit of the denominator is not zero.

6. *Power Rule:* If r and s are integers with no common factor and $s \neq 0$, then

$$\lim_{x \rightarrow c} (f(x))^{r/s} = L^{r/s}$$

Example.:

$$(a) \lim_{x \rightarrow c} (x^3 + 4x^2 - 3) = \lim_{x \rightarrow c} x^3 + \lim_{x \rightarrow c} 4x^2 - \lim_{x \rightarrow c} 3 \quad \text{Sum and Difference Rules}$$

$$= c^3 + 4c^2 - 3 \quad \text{Product and Multiple Rules}$$

$$(b) \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \rightarrow c} (x^4 + x^2 - 1)}{\lim_{x \rightarrow c} (x^2 + 5)} \quad \text{Quotient Rule}$$

$$= \frac{\lim_{x \rightarrow c} x^4 + \lim_{x \rightarrow c} x^2 - \lim_{x \rightarrow c} 1}{\lim_{x \rightarrow c} x^2 + \lim_{x \rightarrow c} 5} \quad \text{Sum and Difference Rules}$$

$$= \frac{c^4 + c^2 - 1}{c^2 + 5} \quad \text{Power or Product Rule}$$

$$(c) \lim_{x \rightarrow -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \rightarrow -2} (4x^2 - 3)} \quad \text{Power Rule with } r/s = 1/2$$

$$= \sqrt{\lim_{x \rightarrow -2} 4x^2 - \lim_{x \rightarrow -2} 3} \quad \text{Difference Rule}$$

$$= \sqrt{4(-2)^2 - 3} \quad \text{Product and Multiple Rules}$$

$$= \sqrt{16 - 3}$$

$$= \sqrt{13}$$

Indeterminate quantities .:

$\frac{0}{0}$	,	$\frac{\infty}{\infty}$	,	$0 \cdot \infty$	,	$\infty - \infty$	,	∞^0	,	0^0
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Example.: $\lim_{x \rightarrow -2} \frac{-2x-4}{x^3+2x^2}$

Solution: $\frac{-2 \cdot -2 - 4}{(-2)^3 + 2(-2)^2} = \frac{0}{0} = \infty$, then the solution

$$\lim_{x \rightarrow -2} \frac{-2(x+2)}{x^2(x+2)} = \lim_{x \rightarrow -2} \frac{-2}{x^2} = \frac{-2}{(-2)^2} = \frac{-2}{4} = \frac{-1}{2}$$

Example.: $\lim_{x \rightarrow 2} \frac{x^2-4}{x-2} \quad \frac{0}{0} = \infty$

Solution: $\lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{x-2} = \lim_{x \rightarrow 2} (x+2) = 2+2 = 4$

Example.: $\lim_{x \rightarrow 0} \frac{x}{1 - \sqrt{1-x}} = \frac{0}{0} = \infty$

Solution: $\lim_{x \rightarrow 0} \frac{x(1 + \sqrt{1-x})}{(1 - \sqrt{1-x})(1 + \sqrt{1-x})} = \lim_{x \rightarrow 0} \frac{x(1 + \sqrt{1-x})}{1 - (1-x)}$

$$= \lim_{x \rightarrow 0} \frac{x(1 + \sqrt{1-x})}{1 - 1 + x} = \lim_{x \rightarrow 0} \frac{x(1 + \sqrt{1-x})}{x} = \lim_{x \rightarrow 0} (1 + \sqrt{1-x}) = 2$$

Example.: $\lim_{x \rightarrow 0} \frac{\sqrt{x^2 + 100} - 10}{x^2} = \frac{0}{0} = \infty$

Solution: $\lim_{x \rightarrow 0} \frac{(\sqrt{x^2 + 100} - 10)(\sqrt{x^2 + 100} + 10)}{x^2(\sqrt{x^2 + 100} + 10)}$

$$= \lim_{x \rightarrow 0} \frac{x^2 + 100 - 100}{x^2(\sqrt{x^2 + 100} + 10)} = \lim_{x \rightarrow 0} \frac{x^2}{x^2(\sqrt{x^2 + 100} + 10)}$$

$$= \lim_{x \rightarrow 0} \frac{1}{\sqrt{x^2 + 100} + 10} = \frac{1}{\sqrt{0 + 100} + 10} = \frac{1}{10 + 10} = \frac{1}{20}$$

Limits of trigonometric functions.:

If a is constant.

$\lim_{x \rightarrow 0} \sin x = 0$	$\lim_{x \rightarrow 0} \cos x = 1$	$\lim_{x \rightarrow 0} \tan x = 0$
$\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$	$\lim_{x \rightarrow 0} \frac{\sin ax}{ax} = 1$	$\lim_{x \rightarrow 0} \frac{\tan x}{x} = 1$
$\lim_{x \rightarrow 0} \frac{\tan ax}{ax} = 1$	$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x} = 0$	

Example.: Find

$$1. \quad \lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{1}{5} \quad \lim_{x \rightarrow 0} \frac{\sin 2x}{x} = \frac{1}{5} \quad \lim_{x \rightarrow 0} \frac{\sin 2x}{x} * \frac{2}{2}$$

$$= \frac{2}{5} \quad \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} = \frac{2}{5} * 1 = \frac{2}{5}$$

$$2. \quad \lim_{x \rightarrow 0} \frac{\sin 2x}{\sin 3x} = \lim_{x \rightarrow 0} \frac{\sin 2x \frac{2x}{2x}}{\sin 3x \frac{3x}{3x}} = \lim_{x \rightarrow 0} \frac{2x \frac{\sin 2x}{2x}}{3x \frac{\sin 3x}{3x}}$$

$$= \frac{2}{3} \quad \lim_{x \rightarrow 0} \frac{\frac{\sin 2x}{2x}}{\frac{\sin 3x}{3x}} = \frac{2}{3} \quad \frac{\lim_{x \rightarrow 0} \frac{\sin 2x}{2x}}{\lim_{x \rightarrow 0} \frac{\sin 3x}{3x}} = \frac{2}{3} * \frac{1}{1} = \frac{2}{3}$$

$$3. \quad \lim_{\theta \rightarrow 0} \frac{\theta}{\sin \theta} = \lim_{\theta \rightarrow 0} \frac{1}{\frac{\sin \theta}{\theta}} = \frac{1}{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta}} = \frac{1}{1} = 1$$

$$4. \quad \lim_{\theta \rightarrow 0} \frac{\sin^2 \theta}{\theta^2} = \left(\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \right)^2 = (1)^2 = 1$$

$$\text{or} \quad \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} * \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 * 1 = 1$$

$$5. \quad \lim_{h \rightarrow 0} \frac{\cos h - 1}{h} = \lim_{h \rightarrow 0} \frac{-2 \sin^2(\frac{h}{2})}{h}$$

$$\text{because } [\sin^2 h = \frac{1 - \cos 2h}{2}, \sin^2(\frac{h}{2}) = \frac{1 - \cos h}{2}]$$

$$= - \lim_{h \rightarrow 0} \frac{\sin^2(\frac{h}{2})}{\frac{h}{2}} = - \lim_{h \rightarrow 0} \frac{\sin(\frac{h}{2})}{\frac{h}{2}} * \lim_{h \rightarrow 0} \sin(\frac{h}{2}) = -1 * 0 = 0$$

$$6. \quad \lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t} = \lim_{t \rightarrow 0} \frac{\frac{\sin t}{\cos t} * \frac{1}{\cos 2t}}{3t}$$

$$= \frac{1}{3} \lim_{t \rightarrow 0} \left(\frac{\sin t}{t} \cdot \frac{1}{\cos t} \cdot \frac{1}{\cos 2t} \right) = \frac{1}{3} \lim_{t \rightarrow 0} \frac{\sin t}{t} \cdot \lim_{t \rightarrow 0} \frac{1}{\cos t} \cdot \lim_{t \rightarrow 0} \frac{1}{\cos 2t}$$

$$= \frac{1}{3} * 1 * 1 * 1 = \frac{1}{3}$$

$$7. \quad \lim_{x \rightarrow 0} \frac{x \sin x}{2 - 2 \cos x} = \lim_{x \rightarrow 0} \frac{(x \sin x)(2 + 2 \cos x)}{(2 - 2 \cos x)(2 + 2 \cos x)}$$

$$= \lim_{x \rightarrow 0} \frac{(x \sin x)(2 + 2 \cos x)}{4 - 4 \cos^2 x} = \lim_{x \rightarrow 0} \frac{(x \sin x)(2 + 2 \cos x)}{4(1 - \cos^2 x)}$$

$$= \lim_{x \rightarrow 0} \frac{(x \sin x)(2 + 2 \cos x)}{4 \sin^2 x} \lim_{x \rightarrow 0} \frac{x(2 + 2 \cos x)}{4 \sin x}$$

$$= \lim_{x \rightarrow 0} \frac{2 + 2 \cos x}{4 \frac{\sin x}{x}} = \frac{2 + 2}{4 * 1} = \frac{4}{4} = 1$$

$$\begin{aligned}
& 8. \lim_{x \rightarrow \frac{\pi}{4}} \frac{\tan x - 1}{x - \frac{\pi}{4}} \quad \left[\text{let } y = x - \frac{\pi}{4}, \quad x \rightarrow \frac{\pi}{4}, \quad y \rightarrow 0 \right] \\
& = \lim_{y \rightarrow 0} \frac{\tan(y + \frac{\pi}{4}) - 1}{y} = \lim_{y \rightarrow 0} \left[\frac{1}{y} (\tan(y + \frac{\pi}{4}) - 1) \right] \\
& \left[\tan(y + \frac{\pi}{4}) = \frac{\tan y + \tan \frac{\pi}{4}}{1 - \tan y \tan \frac{\pi}{4}} = \frac{\tan y + 1}{1 - \tan y} \right] \\
& = \lim_{y \rightarrow 0} \left[\frac{1}{y} * \left(\frac{\tan y + 1}{1 - \tan y} - 1 \right) \right] = \lim_{y \rightarrow 0} \left[\frac{1}{y} \left(\frac{\tan y + 1 - (1 - \tan y)}{1 - \tan y} \right) \right] \\
& = \lim_{y \rightarrow 0} \left[\frac{1}{y} \left(\frac{\tan y + 1 - 1 + \tan y}{1 - \tan y} \right) \right] = \lim_{y \rightarrow 0} \left[\frac{1}{y} \left(\frac{\tan y + \tan y}{1 - \tan y} \right) \right] \\
& = \lim_{y \rightarrow 0} \left[\frac{1}{y} \left(\frac{2 \tan y}{1 - \tan y} \right) \right] = 2 \lim_{y \rightarrow 0} \left[\frac{\tan y}{y} * \frac{1}{1 - \tan y} \right] \\
& = 2 \lim_{y \rightarrow 0} \frac{\tan y}{y} * \lim_{y \rightarrow 0} \frac{1}{1 - \tan y} = 2 * 1 * 1 = 2
\end{aligned}$$

Infinite limits [as $x \rightarrow \pm \infty$] ∴

The basic facts to be verified by applying the formal definition are.

$$\lim_{x \rightarrow \pm \infty} k = k \quad \text{and} \quad \lim_{x \rightarrow \pm \infty} \frac{1}{x} = 0$$

$$\text{Example } \therefore \lim_{x \rightarrow \infty} \left(5 + \frac{1}{x} \right) = \lim_{x \rightarrow \infty} 5 + \lim_{x \rightarrow \infty} \frac{1}{x} = 5 + 0 = 5$$

$$\text{Example } \therefore \lim_{x \rightarrow -\infty} \frac{\pi \sqrt{3}}{x^2} = \lim_{x \rightarrow -\infty} \pi \sqrt{3} \frac{1}{x} * \frac{1}{x} = \lim_{x \rightarrow -\infty} \pi \sqrt{3} *$$

$$\lim_{x \rightarrow -\infty} \frac{1}{x} * \lim_{x \rightarrow -\infty} \frac{1}{x} = \pi \sqrt{3} * 0 * 0 = 0$$

$$\begin{aligned} \text{Example :. } \lim_{x \rightarrow \infty} \frac{5x^2 + 8x - 3}{3x^2 + 2} &= \lim_{x \rightarrow \infty} \frac{\frac{5x^2}{x^2} + \frac{8x}{x^2} - \frac{3}{x^2}}{\frac{3x^2}{x^2} + \frac{2}{x^2}} = \lim_{x \rightarrow \infty} \frac{5 + \frac{8}{x} - \frac{3}{x^2}}{3 + \frac{2}{x^2}} \\ &= \frac{5 + 0 + 0}{3 + 0} = \frac{5}{3} \end{aligned}$$

$$\begin{aligned} \text{Example :. } \lim_{x \rightarrow \infty} \frac{11x + 2}{2x^3 - 1} &= \lim_{x \rightarrow \infty} \frac{\frac{11x}{x^3} + \frac{2}{x^3}}{\frac{2x^3}{x^3} - \frac{1}{x^3}} = \lim_{x \rightarrow \infty} \frac{\frac{11}{x^2} + \frac{2}{x^3}}{2 - \frac{1}{x^3}} \\ &= \frac{0 + 0}{2 - 0} = \frac{0}{2} = 0 \end{aligned}$$

$$\text{Example :. } \lim_{x \rightarrow \infty} \sin\left(\frac{1}{x}\right) = \sin\left(\frac{1}{\infty}\right) = \sin(0) = 0$$

$$\begin{aligned} \text{Example :. } \lim_{x \rightarrow \infty} x \sin\left(\frac{1}{x}\right) &\quad \left[\text{let } t = \frac{1}{x}, x \rightarrow \infty, t \rightarrow 0 \right] \\ &= \lim_{t \rightarrow 0} \frac{1}{t} \sin(0) = \lim_{t \rightarrow 0} \frac{\sin t}{t} = 1 \end{aligned}$$

$$\begin{aligned} \text{Example :. } \lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16}) &= \lim_{x \rightarrow \infty} (x - \sqrt{x^2 + 16}) * \frac{(x + \sqrt{x^2 + 16})}{(x + \sqrt{x^2 + 16})} \\ &= \lim_{x \rightarrow \infty} \frac{x^2 - (x^2 + 16)}{x + \sqrt{x^2 + 16}} = \lim_{x \rightarrow \infty} \frac{x^2 - x^2 - 16}{x + \sqrt{x^2 + 16}} = \lim_{x \rightarrow \infty} \frac{-16}{x + \sqrt{x^2 + 16}} \\ &= \lim_{x \rightarrow \infty} \frac{\frac{-16}{x}}{\frac{x}{x} + \frac{\sqrt{x^2 + 16}}{x}} = \lim_{x \rightarrow \infty} \frac{\frac{-16}{x}}{1 + \sqrt{1 + \frac{16}{x^2}}} = \frac{0}{1 + \sqrt{1 + 0}} = \frac{0}{2} = 0 \end{aligned}$$

H.W: Find .

1. $\lim_{x \rightarrow 2} \frac{x+3}{x+6}$ $(\frac{5}{8})$	2. $\lim_{x \rightarrow 2/3} 3x(2x-1)$ $(\frac{2}{3})$
3. $\lim_{x \rightarrow 5} \frac{x-5}{x^2-25}$ $(\frac{1}{10})$	4. $\lim_{x \rightarrow -3} \frac{x+3}{x^2+4x+3}$ $(\frac{-1}{2})$
5. $\lim_{t \rightarrow 1} \frac{t^2+t-2}{t^2-1}$ $(\frac{3}{2})$	6. $\lim_{u \rightarrow 1} \frac{u^2-1}{u^3-1}$ $(\frac{2}{3})$
7. $\lim_{x \rightarrow 0} (2 \sin x - 1)$ (-1)	8. $\lim_{x \rightarrow 0} \sec x$ (1)
9. $\lim_{x \rightarrow 0} (x^2 - 1)(2 - \cos x)$ (-1)	10. $\lim_{x \rightarrow \pi} \sqrt{x+4} \cos(x+\pi)$ $(\sqrt{\pi+4})$
11. $\lim_{x \rightarrow 0} \frac{1+x+\sin x}{3 \cos x}$ $(\frac{1}{3})$	12. $\lim_{x \rightarrow 0} \frac{\sin(1-\cos x)}{1-\cos x}$ (1)
13. $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\sin 2\theta}$ $(\frac{1}{2})$	14. $\lim_{\theta \rightarrow 0} \theta \cos \theta$ (0)
15. $\lim_{x \rightarrow \pi} \sin(x - \sin x)$ (0)	16. $\lim_{t \rightarrow 0} \cos(\frac{\pi}{\sqrt{19-3 \sec 2t}})$ $(\frac{1}{\sqrt{2}})$
17. $\lim_{x \rightarrow \infty} \sqrt{\frac{8x^2-3}{2x^2+x}}$ (2)	18. $\lim_{x \rightarrow \infty} (\frac{1-x^3}{x^2+7x})^5$ (∞)
19. $\lim_{x \rightarrow \infty} \frac{2\sqrt{x}+x^{-1}}{3x-7}$ (0)	20. $\lim_{x \rightarrow \infty} \frac{\sqrt[3]{x}-\sqrt[5]{x}}{\sqrt[3]{x}+\sqrt[5]{x}}$ (1)
21. $\lim_{x \rightarrow \infty} \frac{3-x}{\sqrt{4x^2+25}}$ $(1/2)$	22. $\lim_{x \rightarrow \infty} (\frac{3}{x^2} - \cos \frac{1}{x})(1 + \sin \frac{1}{x})$ (-1)