Subject Name: Calculus I

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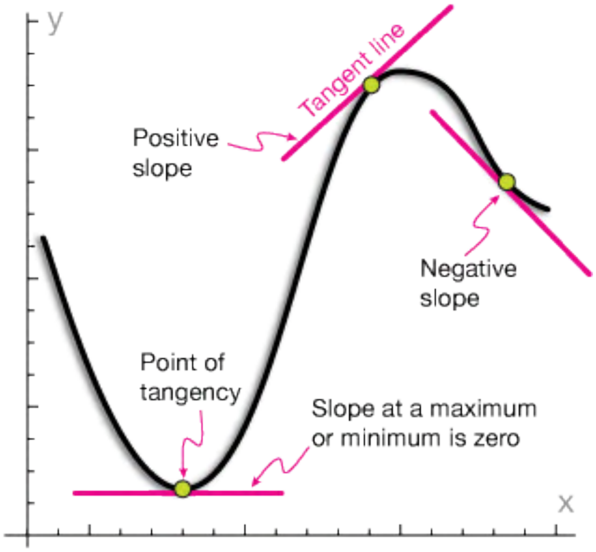
Lecture No. 5

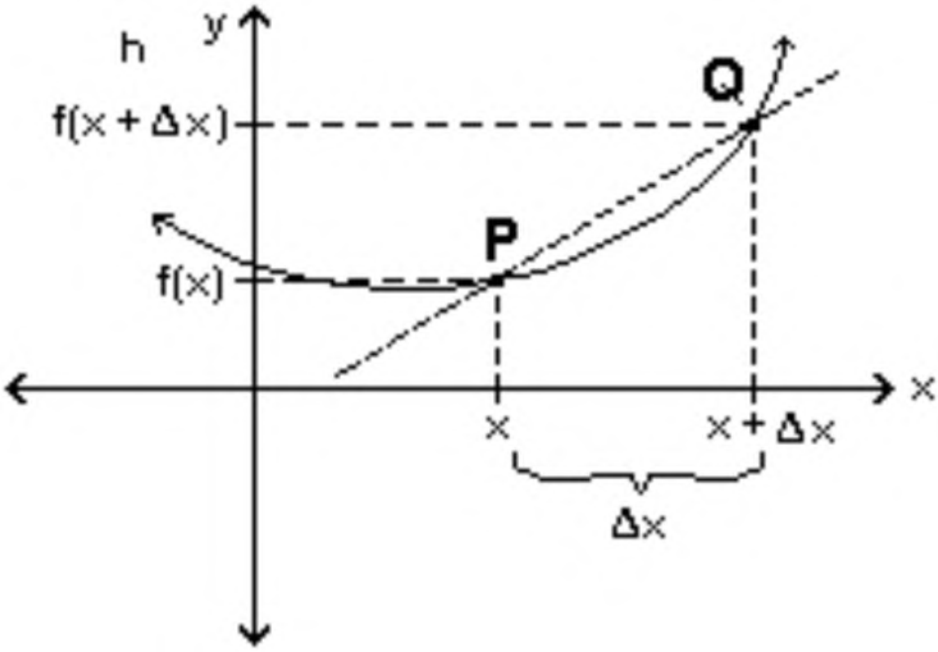
Lecture Title: Differentiation

CHAPTER FOUR: Differentiation

For each point on the curve $y = f(x)$, there is a single straight tangent line at the point; The slop of straight tangent of the curve $y = f(x)$ at the point $(x, f(x))$ represents the derivative at that point.



Let $P(x, f(x))$ be a fixed point on the curve; and $Q(x+\Delta x, f(x+\Delta x))$ be another point, so $\Delta y = f(x + \Delta x) - f(x)$.



Note that: At Δx, decreasing length (close to zero) the straight secant PQ more and more applicability begins on the straight tangent at the point $(x, f(x))$. When $(\Delta x \to 0)$, knowing that the slop straight tangent at the point $(x, f(x))$ represents a derived function at that point.

$$m_{tan} = \lim_{\Delta x \to 0} m_{sec} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$$

$$\boxed{f^{'}(x) = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} \frac{f(x + \Delta x) - f(x)}{\Delta x}}$$

Remark: When the value of the limit exist, the function is called differentiable function, and $f^{'}$ is called the derivative of f at x.

Remark: The equation of the tangent line at a point (x_1, y_1) is given by the following form:

$$\boxed{(y - y_1) = m_{tan}(x - x_1)}$$

Definition: The normal line of a curve is the line that is perpendicular to the tangent of the curve at a particular.

$$\boxed{m_{\perp} = \frac{-1}{m_{tan}}}$$

Remark: The equation of the normal line at a point (x_1, y_1) is given by the following form:

$$\boxed{(y - y_1) = m_{\perp}(x - x_1)}$$

Note $f^{'}(x) = y^{'} = \frac{dy}{dx} = \frac{df(x)}{dx}$

Example 1: Let $f(x) = 4x - 2$, find $f'(x)$ by using the definition?

Solution:-

$f^{'}(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x))}{\Delta x}$

$\because f(x) = 4x - 2,\ f(x + \Delta x) = 4(x + \Delta x) - 2$

$$\begin{aligned} \Longrightarrow f^{'}(x) &= \lim_{\Delta x \to 0} \frac{[4(x + \Delta x) - 2] - [4x - 2])}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{4x + 4\Delta x - 2 - 4x + 2}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{4\Delta x}{\Delta x} \\ &= \lim_{\Delta x \to 0} 4 = 4 \end{aligned}$$

Example 2: Let $f(x) = \sqrt{x}$, find the equation of the tangent line and normal line at the point $(4, 2)$ by using the definition?

Solution:-

We need to find: $m_{tan}]_{(4,2)} = f^{'}(x)]_{(4,2)}$

$$\begin{aligned} \Longrightarrow f^{'}(x) &= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{\sqrt{x + \Delta x} - \sqrt{x}}{\Delta x} . \frac{\sqrt{x + \Delta x} + \sqrt{x}}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \lim_{\Delta x \to 0} \frac{(x + \Delta x) - x}{\Delta x(\sqrt{x + \Delta x} + \sqrt{x})} \\ &= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x + \Delta x} + \sqrt{x}} \\ &= \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}} \end{aligned}$$

$\Longrightarrow m_{tan} = \frac{1}{2\sqrt{x}} \Longrightarrow m_{tan}]_{(4,2)} = f^{'}(x)]_{(4,2)} = \frac{1}{2\sqrt{4}}$

Now, we need to find the equation of the tangent line at the point $(x_1, y_1) = (4, 2)$

$(y - y_1) = m_{tan}(x - x_1)$

$\Longrightarrow y - 2 = \frac{1}{4}(x - 4)$

$\Longrightarrow y = \frac{1}{4}x + 1$

Next, we need to find the equation of the normal line at the point $(x_1, y_1) = (4, 2)$

$\because m_{\perp} = \frac{-1}{m_{tan}} \longrightarrow m_{\perp} = \frac{-1}{\frac{1}{4}} = -4$

$(y - y_1) = m_{\perp}(x - x_1)$

$\implies y - 2 = -4(x - 4)$

$\implies y = -4x + 18$

Problems 4.1:

1. Find $f'(x)$ by using the definition of the following function:-

   (a) $f(x) = x^2$ (b) $f(x) = 4 - \sqrt{x + 3}$

2. Let $f(x) = x^2$, find the equation of the tangent line and normal line at the point $(3, 9)$ by using the definition.

3. Let $f(x) = \sqrt{x + 3}$, find the equation of the tangent line at $x = 2$.

Differentiable VS. Continuous:

Theorem: If f(x) is a differentiable function at x_0, then it is a continuous function at x_0.

Proof: To prove $f(x)$ is continuous function at x_0,

we need to show: $\lim_{x\to 0} f(x) = f(x_0)$ $\left(\text{i.e., } \lim_{x\to 0}[f(x) - f(x_0)] = 0\right)$

Suppose that:

$\Delta x = x - x_0 \implies x = x_0 + \Delta x \implies f(x) = f(x_0 + \Delta x)$

Hence, when $x \to 0$, $\Delta x \to 0$

$$\begin{aligned}
\lim_{x\to 0}[f(x) - f(x_0)] &= \lim_{x\to 0}[f(x_0 + \Delta x) - f(x_0)] \\
&= \lim_{x\to 0}\left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}.\Delta x\right] \\
&= \lim_{x\to 0}\left[\frac{f(x_0 + \Delta x) - f(x_0)}{\Delta x}. \lim_{x\to 0}\Delta x\right] \\
&= f^{'}(x_0).0 = 0
\end{aligned}$$

Note The inverse of the above theorem is not true.

(i.e., If f(x) is a continuous at x_0, then it is not necessary to be differentiable at x_0)

Example: Let $f(x) = |x|$, and $x_0 = 0$.

From the above plot $f(x) = |x|$ is continuous at $x_0 = 0$.

However, $f(x) = |x|$ is **not differentiable** at $x_0 = 0$.

Proof:

$$\because |x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$\therefore |\Delta x| = \begin{cases} \Delta x & \Delta x \geq 0 \\ -\Delta x & \Delta x < 0 \end{cases}$$

$$\begin{aligned} f^{'}(x) &= \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x))}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{|x+\Delta x| - |x|}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{|0+\Delta x| - |0|}{\Delta x} \\ &= \lim_{\Delta x \to 0} \frac{|\Delta x|}{\Delta x} \end{aligned}$$

Hence, $L^{+} = \lim_{\Delta x \to 0^{+}} = 1$ & $L^{-} = \lim_{\Delta x \to 0^{-}} = -1$

Since, $L^{+} \neq L^{-} \implies$ The limit does not exists.

$\therefore f(x)$ is not a differentiable function at $x_0 = 0$