# Filter Design

## 1.1. Introduction

This chapter addresses the problem of designing digital filters. The design process begins with defining the filter specifications, which may include constraints on the magnitude and/or phase of the frequency response, constraints on the unit sample response or step response, determining the type of filter (such as FIR or IIR), and the filter order. Once the specifications are set, the next step is to find a set of filter coefficients that produce an acceptable filter. After the filter has been designed, the final step is to implement the system in hardware or software, quantizing the coefficients if necessary, and choosing an appropriate filter structure.

## 1.2. Filter Specification

Before a filter can be designed, a set of filter specifications must be defined. For example, suppose that we would like to design a low-pass filter with a cutoff frequency $w_c$. The frequency response of an ideal low-pass filter with linear phase and a cutoff frequency $w_c$. is

$$H_d(e^{-j\alpha w}) = \begin{cases} e^{-j\alpha w} & |w| < w_c \\ 0 & w_c < |w| \le \pi \end{cases}$$

As illustrated in Fig. 1-1. Thus, the specifications include the passband cutoff frequency, $w_p$, the stopband cutoff frequency, $w_s$, the passband ripple, $\delta p$ ,. and the stopband attenuation $\delta s$. The passband and stopband deviations are often given in decibels (dB) as follows:

$$\alpha_p = -20\log(1-\delta_p)$$

$$\alpha_s = -20\log(\delta_s)$$

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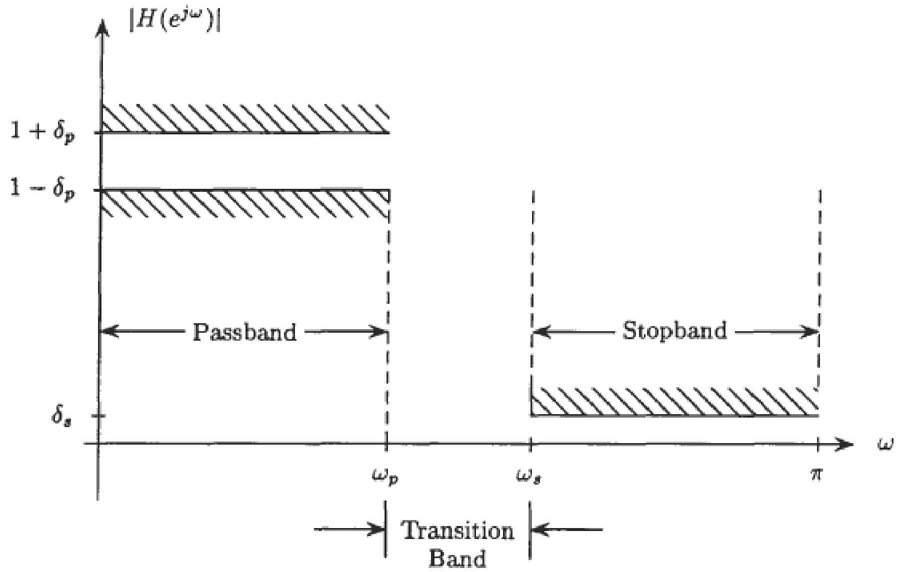


Figure1.1. Filter specifications for a low-pass filter

## 1.3. Finite Impulse Response (FIR) Filter Format

FIR filters have two important advantages over IIR filters. First, they are guaranteed to be stable, even after the filter coefficients have been quantified. Second, they may be easily constrained to have (generalized) linear phase. The frequency response of an N th-order causal FIR filter is:

$$H(e^{j\omega}) = \sum_{n=0}^{N} h(n) e^{-jn\omega}$$

FIR filter is completely specified by the following input-output relationship:

$$y[n] = \sum_{i=0}^{N} b_i x[n-i]$$
$$= b_0 x[n] + b_1 x[n-1] + b_2 x[n-2] + \cdots + b_N x[n-N]$$

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Where $b_i$ represents FIR filter coefficients and N + 1 denotes the FIR filter length. Applying the z-transform on both sides of the above Equation leads to

$$Y(Z) = \sum_{i=0}^{N} b_i X(Z) Z^{-i}$$

$$H(Z) = \sum_{n=-\infty}^{\infty} h[n] Z^{-n}$$

For Non-Causal FIR Filter

$$\mathrm{H(Z))} = \cdots + h[-2]Z^2 + h[-1]Z + h[0] + h[1]Z^{-1} + h[2]Z^{-2} + \cdots$$

For a Causal FIR Filter

$$H(Z) = b_0 + b_1 Z^{-1} + \cdots + b_{2N} Z^{-2N}$$

Where, $b_n = h[n-N] \;\; for \;\; n = 0, 1, \ldots, 2N$

$$Y(Z) = b_0 X(Z) + b_1 Z^{-1} X(Z) + b_2 Z^{-2} X(Z) + \cdots + b_N Z^{-N} X(Z)$$

$$H(Z) = \frac{Y(Z)}{X(Z)} = b_0 + b_1 Z^{-1} + \cdots + b_N Z^{-N}$$

The obtained filter is a non-causal z-transfer function of the FIR filter, since the filter transfer function contains terms with positive powers of z, which in turn means that the filter output depends on the future filter inputs. To remedy the noncausal z- transfer function, we delay the truncated impulse response $h$[n] by N samples to yield the following causal FIR filter:

**Example1:** Given the following FIR filter:

$$y[n] = 0.1x[n] + 0.25x[n-1] + 0.2x[n-2]$$

Determine the transfer function, filter length, nonzero coefficients, and impulse response.

**Solution:**

Applying z-transform on both sides of the difference equation yields

$$Y(z) = 0.1X(z) + 0.25X(z)z^{-1} + 0.2X(z)z^{-2}.$$

Then the transfer function is found to be

$$H(z) = \frac{Y(z)}{X(z)} = 0.1 + 0.25z^{-1} + 0.2z^{-2}.$$

The filter length is $N + 1 = 3$, and the identified coefficients are

$$b_o = 0.1,\ b_1 = 0.25,\ b_2 = 0.2$$

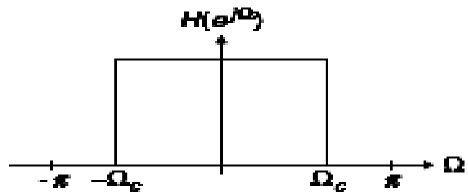
Taking the inverse z-transform of the transfer function

$$h[n] = 0.1\delta[n] + 0.25\delta[n-1] + 0.2\delta[n-2]$$

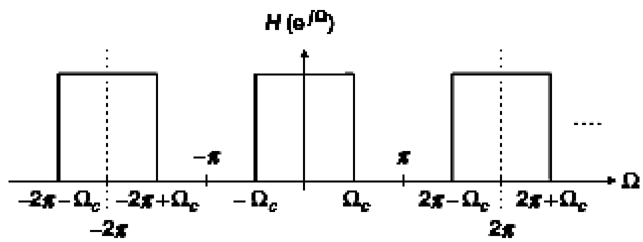
## 1.4. Fourier Transform Design of FIR

An ideal low pass filter with a normalized cutoff frequency $\Omega c$, whose magnitude frequency response in terms of the normalized digital frequency $\Omega$ is plotted in the Figure below and is characterized by

$$H(e^{j\Omega}) = \begin{cases} 1, & 0 \le |\Omega| \le \Omega_c \\ 0, & \Omega_c \le |\Omega| \le \pi. \end{cases}$$




Since the frequency response is periodic with a period of $\Omega = 2\pi$ radians, we can extend the frequency response of the ideal filter $H(e^{j\Omega})$, as shown in the Figure below




The desired impulse response of the ideal filter is

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\Omega})\, e^{j\Omega n} \quad for \quad -\infty < n < \infty$$

The desired impulse response approximation of the ideal lowpass filter is solved as

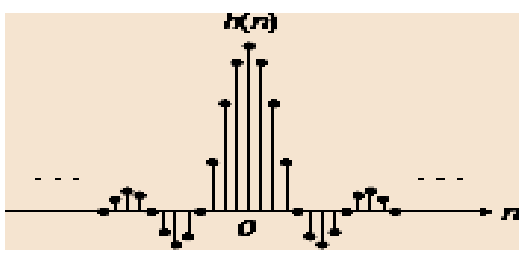
$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\Omega})e^{j\Omega\times 0}d\Omega \qquad for\ n = 0$$

$$h[n] = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\Omega})e^{j\Omega n}d\Omega \qquad for\ n \neq 0$$

$$= \frac{1}{2\pi}\int_{-\Omega_c}^{\Omega_c} H(e^{j\Omega})e^{j\Omega n}d\Omega = \frac{e^{j\Omega n}}{2\pi jn}\Bigg|_{-\Omega_c}^{\Omega_c} = \frac{1}{\pi n}\,\frac{e^{jn\Omega_c} - e^{-jn\Omega_c}}{2j}$$

$$= \frac{\sin(\Omega_c n)}{\pi n}$$

The desired impulse response h(n) is plotted versus the sample number n in the Figure below:

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# Table 1.1. Impulse response for FIR coefficient

| Filter Type | Ideal Impulse response $h[n]$(non causal FIR coefficients) |
|---|---|
| Low Pass | $h[n] = \begin{cases} \frac{\Omega_c}{\pi} & n = 0 \\ \frac{\sin(\Omega_c n)}{\pi n} & -N \le n \le N \end{cases}$ |
| High Pass | $h[n] = \begin{cases} \frac{\pi - \Omega_c}{\pi} & n = 0 \\ -\frac{\sin(\Omega_c n)}{\pi n} & -N \le n \le N \end{cases}$ |
| Band Pass | $h[n] = \begin{cases} \frac{\Omega_H - \Omega_L}{\pi} & n = 0 \\ \frac{\sin(\Omega_H n)}{\pi n} - \frac{\sin(\Omega_L n)}{\pi n} & -N \le n \le N \end{cases}$ |
| Band Stop | $h[n] = \begin{cases} \frac{\pi - \Omega_H + \Omega_L}{\pi} & n = 0 \\ -\frac{\sin(\Omega_H n)}{\pi n} + \frac{\sin(\Omega_L n)}{\pi n} & -N \le n \le N \end{cases}$ |

**Example2: a)** Calculate the filter coefficients for a 3-tap FIR **low pass** filter with a cutoff frequency of 800 Hz and a sampling rate of 8,000 Hz using the Fourier transform method. **b)** Determine the transfer function and difference equation of the designed FIR system. c) Compute and plot the magnitude frequency response for: $\Omega = 0, \frac{\pi}{4}, \frac{\pi}{2}, \frac{3\pi}{4}, and\ \pi\ radians.$

Solution:

a. Calculating the normalized cutoff frequency leads to

$$\Omega_c = 2\pi f_c T_s = 2\pi \times \frac{800}{8000} = 0.2\pi\ rad$$

Since in this case 2N+1=3, using the equation:

$$h[n] = \begin{cases} \dfrac{\Omega_c}{\pi} & n = 0 \\ \dfrac{\sin(\Omega_c n)}{\pi n} & -N \le n \le N \end{cases}$$

The computed filter coefficients via the previous expression are listed as:

$$h[0] = \frac{0.2\pi}{\pi} = 0.2$$

$$h[1] = \frac{\sin(0.2\pi)}{\pi} = 0.1871$$

Using the symmetry leads to

$$h[-1] = h[1] = 0.1871$$

Thus delaying h(n) by $N = 1$ sample

$$b_0 = h[0-1] = h[-1] = 0.1871$$

$$b_1 = h[1-1] = h[0] = 0.2$$

$$b_2 = h[2-1] = h[1] = 0.1871$$

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b. The transfer function is achieved as:

$$H(z) = 0.1871 + 0.2z^{-1} + 0.1871z^{-2}.$$

$$\frac{Y(z)}{X(z)} = H(z) = 0.1871 + 0.2z^{-1} + 0.1817z^{-2}.$$

$$Y(z) = 0.1871X(z) + 0.2z^{-1}X(z) + 0.1871z^{-2}X(z).$$

Applying the inverse z-transform on both sides, the difference equation is yielded as

$$y[n] = 0.1871x[n] + 0.2x[n-1] + 0.1871x[n-2]$$

c. The magnitude frequency response and phase response can be obtained by substituting $Z = e^{j\Omega}$ into $H(Z)$

$$H(e^{j\Omega}) = 0.1871 + 0.2e^{-j\Omega} + 0.1871e^{-j2\Omega}.$$

Factoring the term $e^{-j\Omega}$ and using the Euler formula $e^{jx} + e^{-jx} = 2\cos(x)$, we achieve

$$H(e^{j\Omega}) = e^{-j\Omega}(0.1871e^{j\Omega} + 0.2 + 0.1871e^{-j\Omega})$$
$$= e^{-j\Omega}(0.2 + 0.3742\cos(\Omega))$$

Then the magnitude frequency response and phase response are found to be

$$|H(e^{j\Omega})| = |0.2 + 0.3472\cos\Omega|$$

$$\text{and } \angle H(e^{j\Omega}) = \begin{cases} -\Omega & \text{if } 0.2 + 0.3472\cos\Omega > 0 \\ -\Omega + \pi & \text{if } 0.2 + 0.3472\cos\Omega < 0. \end{cases}$$

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