



8.5 RATIONAL FUNCTIONS

A **rational function** is a fraction with polynomials in the numerator and denominator. For example,

$$\frac{x^3}{x^2 + x - 6}, \quad \frac{1}{(x - 3)^2}, \quad \frac{x^2 + 1}{x^2 - 1},$$

are all rational functions of x . There is a general technique called “partial fractions” that, in principle, allows us to integrate any rational function. The algebraic steps in the technique are rather cumbersome if the polynomial in the denominator has degree more than 2, and the technique requires that we factor the denominator, something that is not always possible. However, in practice one does not often run across rational functions with high degree polynomials in the denominator for which one has to find the antiderivative function. So we shall explain how to find the antiderivative of a rational function only when the denominator is a quadratic polynomial $ax^2 + bx + c$.

We should mention a special type of rational function that we already know how to integrate: If the denominator has the form $(ax + b)^n$, the substitution $u = ax + b$ will always work. The denominator becomes u^n , and each x in the numerator is replaced by $(u - b)/a$, and $dx = du/a$. While it may be tedious to complete the integration if the numerator has high degree, it is merely a matter of algebra.



EXAMPLE 8.5.1 Find $\int \frac{x^3}{(3-2x)^5} dx$. Using the substitution $u = 3 - 2x$ we get

$$\begin{aligned}\int \frac{x^3}{(3-2x)^5} dx &= \frac{1}{-2} \int \frac{\left(\frac{u-3}{-2}\right)^3}{u^5} du = \frac{1}{16} \int \frac{u^3 - 9u^2 + 27u - 27}{u^5} du \\&= \frac{1}{16} \int u^{-2} - 9u^{-3} + 27u^{-4} - 27u^{-5} du \\&= \frac{1}{16} \left(\frac{u^{-1}}{-1} - \frac{9u^{-2}}{-2} + \frac{27u^{-3}}{-3} - \frac{27u^{-4}}{-4} \right) + C \\&= \frac{1}{16} \left(\frac{(3-2x)^{-1}}{-1} - \frac{9(3-2x)^{-2}}{-2} + \frac{27(3-2x)^{-3}}{-3} - \frac{27(3-2x)^{-4}}{-4} \right) + C \\&= -\frac{1}{16(3-2x)} + \frac{9}{32(3-2x)^2} - \frac{9}{16(3-2x)^3} + \frac{27}{64(3-2x)^4} + C\end{aligned}$$

□

We now proceed to the case in which the denominator is a quadratic polynomial. We can always factor out the coefficient of x^2 and put it outside the integral, so we can assume that the denominator has the form $x^2 + bx + c$. There are three possible cases, depending on how the quadratic factors: either $x^2 + bx + c = (x-r)(x-s)$, $x^2 + bx + c = (x-r)^2$, or it doesn't factor. We can use the quadratic formula to decide which of these we have, and to factor the quadratic if it is possible.

EXAMPLE 8.5.2 Determine whether $x^2 + x + 1$ factors, and factor it if possible. The quadratic formula tells us that $x^2 + x + 1 = 0$ when

$$x = \frac{-1 \pm \sqrt{1-4}}{2}.$$

Since there is no square root of -3 , this quadratic does not factor.

□

EXAMPLE 8.5.3 Determine whether $x^2 - x - 1$ factors, and factor it if possible. The quadratic formula tells us that $x^2 - x - 1 = 0$ when

$$x = \frac{1 \pm \sqrt{1+4}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

Therefore

$$x^2 - x - 1 = \left(x - \frac{1+\sqrt{5}}{2}\right) \left(x - \frac{1-\sqrt{5}}{2}\right).$$

□



If $x^2 + bx + c = (x - r)^2$ then we have the special case we have already seen, that can be handled with a substitution. The other two cases require different approaches.

If $x^2 + bx + c = (x - r)(x - s)$, we have an integral of the form

$$\int \frac{p(x)}{(x - r)(x - s)} dx$$

where $p(x)$ is a polynomial. The first step is to make sure that $p(x)$ has degree less than 2.

EXAMPLE 8.5.4 Rewrite $\int \frac{x^3}{(x - 2)(x + 3)} dx$ in terms of an integral with a numerator that has degree less than 2. To do this we use long division of polynomials to discover that

$$\frac{x^3}{(x - 2)(x + 3)} = \frac{x^3}{x^2 + x - 6} = x - 1 + \frac{7x - 6}{x^2 + x - 6} = x - 1 + \frac{7x - 6}{(x - 2)(x + 3)},$$

so

$$\int \frac{x^3}{(x - 2)(x + 3)} dx = \int x - 1 dx + \int \frac{7x - 6}{(x - 2)(x + 3)} dx.$$

The first integral is easy, so only the second requires some work. □

Now consider the following simple algebra of fractions:

$$\frac{A}{x - r} + \frac{B}{x - s} = \frac{A(x - s) + B(x - r)}{(x - r)(x - s)} = \frac{(A + B)x - As - Br}{(x - r)(x - s)}.$$

That is, adding two fractions with constant numerator and denominators $(x - r)$ and $(x - s)$ produces a fraction with denominator $(x - r)(x - s)$ and a polynomial of degree less than 2 for the numerator. We want to reverse this process: starting with a single fraction, we want to write it as a sum of two simpler fractions. An example should make it clear how to proceed.

EXAMPLE 8.5.5 Evaluate $\int \frac{x^3}{(x - 2)(x + 3)} dx$. We start by writing $\frac{7x - 6}{(x - 2)(x + 3)}$ as the sum of two fractions. We want to end up with

$$\frac{7x - 6}{(x - 2)(x + 3)} = \frac{A}{x - 2} + \frac{B}{x + 3}.$$

If we go ahead and add the fractions on the right hand side we get

$$\frac{7x - 6}{(x - 2)(x + 3)} = \frac{(A + B)x + 3A - 2B}{(x - 2)(x + 3)}.$$

So all we need to do is find A and B so that $7x - 6 = (A + B)x + 3A - 2B$, which is to say, we need $7 = A + B$ and $-6 = 3A - 2B$. This is a problem you've seen before: solve a



system of two equations in two unknowns. There are many ways to proceed; here's one: If $7 = A + B$ then $B = 7 - A$ and so $-6 = 3A - 2B = 3A - 2(7 - A) = 3A - 14 + 2A = 5A - 14$. This is easy to solve for A : $A = 8/5$, and then $B = 7 - A = 7 - 8/5 = 27/5$. Thus

$$\int \frac{7x - 6}{(x - 2)(x + 3)} dx = \int \frac{8}{5} \frac{1}{x - 2} + \frac{27}{5} \frac{1}{x + 3} dx = \frac{8}{5} \ln |x - 2| + \frac{27}{5} \ln |x + 3| + C.$$

The answer to the original problem is now

$$\begin{aligned} \int \frac{x^3}{(x - 2)(x + 3)} dx &= \int x - 1 dx + \int \frac{7x - 6}{(x - 2)(x + 3)} dx \\ &= \frac{x^2}{2} - x + \frac{8}{5} \ln |x - 2| + \frac{27}{5} \ln |x + 3| + C. \end{aligned}$$

□

Now suppose that $x^2 + bx + c$ doesn't factor. Again we can use long division to ensure that the numerator has degree less than 2, then we complete the square.

EXAMPLE 8.5.6 Evaluate $\int \frac{x + 1}{x^2 + 4x + 8} dx$. The quadratic denominator does not factor. We could complete the square and use a trigonometric substitution, but it is simpler to rearrange the integrand:

$$\int \frac{x + 1}{x^2 + 4x + 8} dx = \int \frac{x + 2}{x^2 + 4x + 8} dx - \int \frac{1}{x^2 + 4x + 8} dx.$$

The first integral is an easy substitution problem, using $u = x^2 + 4x + 8$:

$$\int \frac{x + 2}{x^2 + 4x + 8} dx = \frac{1}{2} \int \frac{du}{u} = \frac{1}{2} \ln |x^2 + 4x + 8|.$$

For the second integral we complete the square:

$$x^2 + 4x + 8 = (x + 2)^2 + 4 = 4 \left(\left(\frac{x + 2}{2} \right)^2 + 1 \right),$$

making the integral

$$\frac{1}{4} \int \frac{1}{\left(\frac{x + 2}{2} \right)^2 + 1} dx.$$

Using $u = \frac{x + 2}{2}$ we get

$$\frac{1}{4} \int \frac{1}{\left(\frac{x + 2}{2} \right)^2 + 1} dx = \frac{1}{4} \int \frac{2}{u^2 + 1} du = \frac{1}{2} \arctan \left(\frac{x + 2}{2} \right).$$

The final answer is now

$$\int \frac{x + 1}{x^2 + 4x + 8} dx = \frac{1}{2} \ln |x^2 + 4x + 8| - \frac{1}{2} \arctan \left(\frac{x + 2}{2} \right) + C.$$

□

Exercises 8.5.

Find the antiderivatives.

1. $\int \frac{1}{4-x^2} dx \Rightarrow$

3. $\int \frac{1}{x^2+10x+25} dx \Rightarrow$

5. $\int \frac{x^4}{4+x^2} dx \Rightarrow$

7. $\int \frac{x^3}{4+x^2} dx \Rightarrow$

9. $\int \frac{1}{2x^2-x-3} dx \Rightarrow$

2. $\int \frac{x^4}{4-x^2} dx \Rightarrow$

4. $\int \frac{x^2}{4-x^2} dx \Rightarrow$

6. $\int \frac{1}{x^2+10x+29} dx \Rightarrow$

8. $\int \frac{1}{x^2+10x+21} dx \Rightarrow$

10. $\int \frac{1}{x^2+3x} dx \Rightarrow$

8.6 NUMERICAL INTEGRATION

We have now seen some of the most generally useful methods for discovering antiderivatives, and there are others. Unfortunately, some functions have no simple antiderivatives; in such cases if the value of a definite integral is needed it will have to be approximated. We will see two methods that work reasonably well and yet are fairly simple; in some cases more sophisticated techniques will be needed.

Of course, we already know one way to approximate an integral: if we think of the integral as computing an area, we can add up the areas of some rectangles. While this is quite simple, it is usually the case that a large number of rectangles is needed to get acceptable accuracy. A similar approach is much better: we approximate the area under a curve over a small interval as the area of a trapezoid. In figure 8.6.1 we see an area under a curve approximated by rectangles and by trapezoids; it is apparent that the trapezoids give a substantially better approximation on each subinterval.

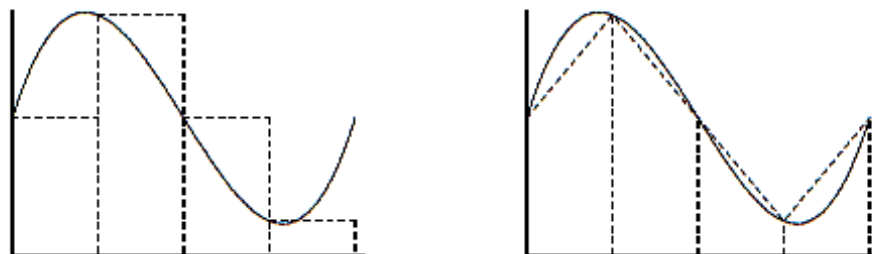


Figure 8.6.1 Approximating an area with rectangles and with trapezoids. (AP)

As with rectangles, we divide the interval into n equal subintervals of length Δx . A typical trapezoid is pictured in figure 8.6.2; it has area $\frac{f(x_i) + f(x_{i+1})}{2} \Delta x$. If we add up

the areas of all trapezoids we get

$$\frac{f(x_0) + f(x_1)}{2} \Delta x + \frac{f(x_1) + f(x_2)}{2} \Delta x + \cdots + \frac{f(x_{n-1}) + f(x_n)}{2} \Delta x =$$

$$\left(\frac{f(x_0)}{2} + f(x_1) + f(x_2) + \cdots + f(x_{n-1}) + \frac{f(x_n)}{2} \right) \Delta x.$$

This is usually known as the **Trapezoid Rule**. For a modest number of subintervals this is not too difficult to do with a calculator; a computer can easily do many subintervals.

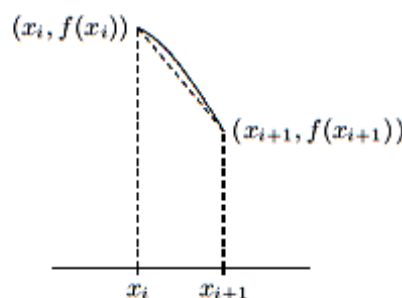


Figure 8.6.2 A single trapezoid.

In practice, an approximation is useful only if we know how accurate it is; for example, we might need a particular value accurate to three decimal places. When we compute a particular approximation to an integral, the error is the difference between the approximation and the true value of the integral. For any approximation technique, we need an **error estimate**, a value that is guaranteed to be larger than the actual error. If A is an approximation and E is the associated error estimate, then we know that the true value of the integral is between $A - E$ and $A + E$. In the case of our approximation of the integral, we want $E = E(\Delta x)$ to be a function of Δx that gets small rapidly as Δx gets small. Fortunately, for many functions, there is such an error estimate associated with the trapezoid approximation.

THEOREM 8.6.1 Suppose f has a second derivative f'' everywhere on the interval $[a, b]$, and $|f''(x)| \leq M$ for all x in the interval. With $\Delta x = (b - a)/n$, an error estimate for the trapezoid approximation is

$$E(\Delta x) = \frac{b-a}{12} M (\Delta x)^2 = \frac{(b-a)^3}{12n^2} M.$$

■

Let's see how we can use this.



EXAMPLE 8.6.2 Approximate $\int_0^1 e^{-x^2} dx$ to two decimal places. The second derivative of $f = e^{-x^2}$ is $(4x^2 - 2)e^{-x^2}$, and it is not hard to see that on $[0, 1]$, $|(4x^2 - 2)e^{-x^2}| \leq 2$. We begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need $E(\Delta x) < 0.005$ or

$$\begin{aligned}\frac{1}{12}(2)\frac{1}{n^2} &< 0.005 \\ \frac{1}{6}(200) &< n^2 \\ 5.77 &\approx \sqrt{\frac{100}{3}} < n\end{aligned}$$

With $n = 6$, the error estimate is thus $1/6^3 < 0.0047$. We compute the trapezoid approximation for six intervals:

$$\left(\frac{f(0)}{2} + f(1/6) + f(2/6) + \cdots + f(5/6) + \frac{f(1)}{2}\right)\frac{1}{6} \approx 0.74512.$$

So the true value of the integral is between $0.74512 - 0.0047 = 0.74042$ and $0.74512 + 0.0047 = 0.74982$. Unfortunately, the first rounds to 0.74 and the second rounds to 0.75, so we can't be sure of the correct value in the second decimal place; we need to pick a larger n . As it turns out, we need to go to $n = 12$ to get two bounds that both round to the same value, which turns out to be 0.75. For comparison, using 12 rectangles to approximate the area gives 0.7727, which is considerably less accurate than the approximation using six trapezoids.

In practice it generally pays to start by requiring better than the maximum possible error; for example, we might have initially required $E(\Delta x) < 0.001$, or

$$\begin{aligned}\frac{1}{12}(2)\frac{1}{n^2} &< 0.001 \\ \frac{1}{6}(1000) &< n^2 \\ 12.91 &\approx \sqrt{\frac{500}{3}} < n\end{aligned}$$

Had we immediately tried $n = 13$ this would have given us the desired answer. \square

The trapezoid approximation works well, especially compared to rectangles, because the tops of the trapezoids form a reasonably good approximation to the curve when Δx is fairly small. We can extend this idea: what if we try to approximate the curve more closely,



by using something other than a straight line? The obvious candidate is a parabola: if we can approximate a short piece of the curve with a parabola with equation $y = ax^2 + bx + c$, we can easily compute the area under the parabola.

There are an infinite number of parabolas through any two given points, but only one through three given points. If we find a parabola through three consecutive points $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, $(x_{i+2}, f(x_{i+2}))$ on the curve, it should be quite close to the curve over the whole interval $[x_i, x_{i+2}]$, as in figure 8.6.3. If we divide the interval $[a, b]$ into an even number of subintervals, we can then approximate the curve by a sequence of parabolas, each covering two of the subintervals. For this to be practical, we would like a simple formula for the area under one parabola, namely, the parabola through $(x_i, f(x_i))$, $(x_{i+1}, f(x_{i+1}))$, and $(x_{i+2}, f(x_{i+2}))$. That is, we should attempt to write down the parabola $y = ax^2 + bx + c$ through these points and then integrate it, and hope that the result is fairly simple. Although the algebra involved is messy, this turns out to be possible. The algebra is well within the capability of a good computer algebra system like Sage, so we will present the result without all of the algebra; you can see how to do it in this Sage worksheet.

To find the parabola, we solve these three equations for a , b , and c :

$$\begin{aligned}f(x_i) &= a(x_{i+1} - \Delta x)^2 + b(x_{i+1} - \Delta x) + c \\f(x_{i+1}) &= a(x_{i+1})^2 + b(x_{i+1}) + c \\f(x_{i+2}) &= a(x_{i+1} + \Delta x)^2 + b(x_{i+1} + \Delta x) + c\end{aligned}$$

Not surprisingly, the solutions turn out to be quite messy. Nevertheless, Sage can easily compute and simplify the integral to get

$$\int_{x_{i+1}-\Delta x}^{x_{i+1}+\Delta x} ax^2 + bx + c dx = \frac{\Delta x}{3}(f(x_i) + 4f(x_{i+1}) + f(x_{i+2})).$$

Now the sum of the areas under all parabolas is

$$\begin{aligned}\frac{\Delta x}{3}(f(x_0) + 4f(x_1) + f(x_2) + f(x_2) + 4f(x_3) + f(x_4) + \cdots + f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)) = \\ \frac{\Delta x}{3}(f(x_0) + 4f(x_1) + 2f(x_2) + 4f(x_3) + 2f(x_4) + \cdots + 2f(x_{n-2}) + 4f(x_{n-1}) + f(x_n)).\end{aligned}$$

This is just slightly more complicated than the formula for trapezoids; we need to remember the alternating 2 and 4 coefficients; note that n must be even for this to make sense. This approximation technique is referred to as **Simpson's Rule**.

As with the trapezoid method, this is useful only with an error estimate:

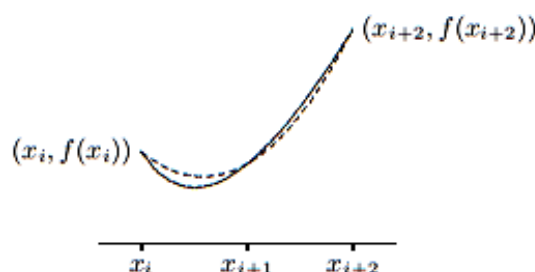


Figure 8.6.3 A parabola (dashed) approximating a curve (solid). (AP)

THEOREM 8.6.3 Suppose f has a fourth derivative $f^{(4)}$ everywhere on the interval $[a, b]$, and $|f^{(4)}(x)| \leq M$ for all x in the interval. With $\Delta x = (b - a)/n$, an error estimate for Simpson's approximation is

$$E(\Delta x) = \frac{b-a}{180} M (\Delta x)^4 = \frac{(b-a)^5}{180n^4} M.$$

■

EXAMPLE 8.6.4 Let us again approximate $\int_0^1 e^{-x^2} dx$ to two decimal places. The fourth derivative of $f = e^{-x^2}$ is $(16x^4 - 48x^2 + 12)e^{-x^2}$; on $[0, 1]$ this is at most 12 in absolute value. We begin by estimating the number of subintervals we are likely to need. To get two decimal places of accuracy, we will certainly need $E(\Delta x) < 0.005$, but taking a cue from our earlier example, let's require $E(\Delta x) < 0.001$:

$$\begin{aligned} \frac{1}{180}(12)\frac{1}{n^4} &< 0.001 \\ \frac{200}{3} &< n^4 \\ 2.86 \approx \sqrt[4]{\frac{200}{3}} &< n \end{aligned}$$

So we try $n = 4$, since we need an even number of subintervals. Then the error estimate is $12/180/4^4 < 0.0003$ and the approximation is

$$(f(0) + 4f(1/4) + 2f(1/2) + 4f(3/4) + f(1))\frac{1}{3 \cdot 4} \approx 0.746855.$$

So the true value of the integral is between $0.746855 - 0.0003 = 0.746555$ and $0.746855 + 0.0003 = 0.7471555$, both of which round to 0.75. □



Exercises 8.6.

In the following problems, compute the trapezoid and Simpson approximations using 4 subintervals, and compute the error estimate for each. (Finding the maximum values of the second and fourth derivatives can be challenging for some of these; you may use a graphing calculator or computer software to estimate the maximum values.) If you have access to Sage or similar software, approximate each integral to two decimal places. You can use this Sage worksheet to get started.

1. $\int_1^3 x \, dx \Rightarrow$

2. $\int_0^3 x^2 \, dx \Rightarrow$

3. $\int_2^4 x^3 \, dx \Rightarrow$

4. $\int_1^3 \frac{1}{x} \, dx \Rightarrow$

5. $\int_1^2 \frac{1}{1+x^2} \, dx \Rightarrow$

6. $\int_0^1 x\sqrt{1+x} \, dx \Rightarrow$

7. $\int_1^5 \frac{x}{1+x} \, dx \Rightarrow$

8. $\int_0^1 \sqrt{x^3+1} \, dx \Rightarrow$

9. $\int_0^1 \sqrt{x^4+1} \, dx \Rightarrow$

10. $\int_1^4 \sqrt{1+1/x} \, dx \Rightarrow$

11. Using Simpson's rule on a parabola $f(x)$, even with just two subintervals, gives the exact value of the integral, because the parabolas used to approximate f will be f itself. Remarkably, Simpson's rule also computes the integral of a cubic function $f(x) = ax^3 + bx^2 + cx + d$ exactly. Show this is true by showing that

$$\int_{x_0}^{x_2} f(x) \, dx = \frac{x_2 - x_0}{3 \cdot 2} (f(x_0) + 4f((x_0 + x_2)/2) + f(x_2)).$$

Note that the right hand side of this equation is exactly the Simpson approximation for the cubic. This does require a bit of messy algebra, so you may prefer to use Sage.

8.7 ADDITIONAL EXERCISES

These problems require the techniques of this chapter, and are in no particular order. Some problems may be done in more than one way.

1. $\int (t+4)^3 \, dt \Rightarrow$

2. $\int t(t^2-9)^{3/2} \, dt \Rightarrow$

3. $\int (e^{t^2} + 16)te^{t^2} \, dt \Rightarrow$

4. $\int \sin t \cos 2t \, dt \Rightarrow$

5. $\int \tan t \sec^2 t \, dt \Rightarrow$

6. $\int \frac{2t+1}{t^2+t+3} \, dt \Rightarrow$

7. $\int \frac{1}{t(t^2-4)} \, dt \Rightarrow$

8. $\int \frac{1}{(25-t^2)^{3/2}} \, dt \Rightarrow$

9. $\int \frac{\cos 3t}{\sqrt{\sin 3t}} \, dt \Rightarrow$

10. $\int t \sec^2 t \, dt \Rightarrow$

11. $\int \frac{e^t}{\sqrt{e^t+1}} \, dt \Rightarrow$

12. $\int \cos^4 t \, dt \Rightarrow$



13. $\int \frac{1}{t^2 + 3t} dt \Rightarrow$

15. $\int \frac{\sec^2 t}{(1 + \tan t)^3} dt \Rightarrow$

17. $\int e^t \sin t dt \Rightarrow$

19. $\int \frac{t^3}{(2 - t^2)^{5/2}} dt \Rightarrow$

21. $\int \frac{\arctan 2t}{1 + 4t^2} dt \Rightarrow$

23. $\int \sin^3 t \cos^4 t dt \Rightarrow$

25. $\int \frac{1}{t(\ln t)^2} dt \Rightarrow$

27. $\int t^3 e^t dt \Rightarrow$

14. $\int \frac{1}{t^2 \sqrt{1 + t^2}} dt \Rightarrow$

16. $\int t^3 \sqrt{t^2 + 1} dt \Rightarrow$

18. $\int (t^{3/2} + 47)^3 \sqrt{t} dt \Rightarrow$

20. $\int \frac{1}{t(9 + 4t^2)} dt \Rightarrow$

22. $\int \frac{t}{t^2 + 2t - 3} dt \Rightarrow$

24. $\int \frac{1}{t^2 - 6t + 9} dt \Rightarrow$

26. $\int t(\ln t)^2 dt \Rightarrow$

28. $\int \frac{t + 1}{t^2 + t - 1} dt \Rightarrow$