

Function of Limits ::

دالة الغايات
أو دالة النهايات

- Limits describe how a function behaves near a point.

In other words;

- The limits are defined as the value that the function approaches as it goes to an x value

تعريف النهاية: x يأخذ قيمة الدالة عند اقتراب x من قيمة المتغير القريب x .

What is limit in calculus in real life ::

ما هو النهايات في الحساب؟ كيان الحقيقة

The real life limits are used any time. where measuring the temperature of an ice cube immersed/sunk in a warm glass of water is a limit, this is one example. other example is measuring of an electric is a limit, magnets or gravitational fields are all limits. All are reaches or approaches a steady solution.

- A Judgement day is the approaching day that the whole live is going to!!

Ex①: Find $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2}$

Solution

نوجد عند اقتراب x من 2 قيمة الدالة عند $x \rightarrow 2$ ونلاحظ بالقرينة ان الدالة غير معرفة عند $x = 2$ لان $2 - 2 = 0$.

① direct substitution =

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \frac{2^2 - 4}{2 - 2} = \frac{0}{0} = \text{undefined}$$



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هذا هو جواب السؤال يجب ان نلاحظ ان الدالة غير معرفة عند $x = 2$.



② Some times we can plug in a value of x that is close to 2 , if not exactly 2 .

- let take $x = 2.1$

$$\therefore f(2.1) = \frac{(2.1)^2 - 4}{2.1 - 2} = \frac{0.41}{0.1} = \underline{4.1}$$

- let take another value that is closer to 2 .

- let take $x = 2.01$

$$f(2.01) = \frac{(2.01)^2 - 4}{2.01 - 2} = \underline{4.01}$$

Note that as we get closer & closer to 2 the limit approaches 4

So, we can say that $\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \underline{4}$

⊛ This technique works to any limit, as long as we plug in a number that's very close to whatever the limit approach to a number is, but not exactly the number & if the limit exists is going to converge to a certain value!!

③ However, there are another techniques to get the answers

lets solve the same example by,

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x-2)(x+2)}{(x-2)} = \lim_{x \rightarrow 2} (x+2) = 2+2 = \boxed{4}$$



Ex (2): $\lim_{x \rightarrow 5} (x^2 + 2x - 4)$

Solution

Here we have no fraction that may equal to zero, so, we can always use the direct substitution

$$\lim_{x \rightarrow 5} (x^2 + 2x - 4) = 5^2 + 2(5) - 4 = 35 - 4 = \boxed{31}$$

Ex (3): $\lim_{x \rightarrow 3} \frac{x^3 - 27}{x - 3}$

Solution

Here we can't use the direct substitution, as the fraction will be " $\frac{0}{0}$ = undefined solution". So, in such a case we may think about using a difference of cubes, like;

$$A^3 - B^3 = (A - B)(A^2 + AB + B^2)$$

Compare this expression with what we have in the Numerator "numli"

$$A^3 = x^3$$

$$B^3 = 27$$

$$\begin{aligned} \therefore \lim_{x \rightarrow 3} \frac{x^3 - 27}{x - 3} &= \lim_{x \rightarrow 3} \frac{(x - 3)(x^2 + 3x + 9)}{(x - 3)} \\ &= \lim_{x \rightarrow 3} (x^2 + 3x + 9) \end{aligned}$$

Here we can use a direct substitution

$$\lim_{x \rightarrow 3} (x^2 + 3x + 9) = 3^2 + 3(3) + 9 = \boxed{27}$$



Ex (4) $\lim_{x \rightarrow 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3}$

Solution

In such problems (having a complex fraction), we need to multiply the numerator & the denominator by a common denominator of the two fraction $\frac{1}{x}$ & $\frac{1}{3}$ which is '3x'

$$\begin{aligned}\lim_{x \rightarrow 3} \frac{\frac{1}{x} - \frac{1}{3}}{x-3} \times \frac{3x}{3x} &= \lim_{x \rightarrow 3} \frac{3-x}{3x(x-3)} \\ &= \lim_{x \rightarrow 3} \frac{-(x-3)}{3x(x-3)} \\ &= \lim_{x \rightarrow 3} \frac{-1}{3x} = \frac{-1}{3(3)} = \boxed{\frac{-1}{9}}\end{aligned}$$

Ex (5) $\lim_{x \rightarrow 9} \frac{\sqrt{x}-3}{x-9}$

Solution

In such case, we need to multiply the top & the bottom by the numerator's conjugate

$$\begin{aligned}\lim_{x \rightarrow 9} \frac{\sqrt{x}-3}{x-9} \times \frac{(\sqrt{x}+3)}{(\sqrt{x}+3)} &= \lim_{x \rightarrow 9} \frac{(x-9)}{(x-9)(\sqrt{x}+3)} \\ &= \lim_{x \rightarrow 9} \frac{1}{\sqrt{x}+3} = \frac{1}{\sqrt{9}+3} = \frac{1}{3+3} \\ &= \boxed{\frac{1}{6}}\end{aligned}$$



Ex ⑥ $\lim_{x \rightarrow 4} \frac{\frac{1}{\sqrt{x}} - \frac{1}{2}}{x-4}$

Solution

In such case, we need to multiply the top & the bottom of the ~~eqs~~ not just by the common denominator but also by the conjugate !.

- Let's start with a common denominator, which is " $2\sqrt{x}$ "

$$\lim_{x \rightarrow 4} \frac{(\frac{1}{\sqrt{x}} - \frac{1}{2})}{(x-4)} \times \frac{2\sqrt{x}}{2\sqrt{x}} = \lim_{x \rightarrow 4} \frac{2 - \sqrt{x}}{2\sqrt{x}(x-4)}$$

- Now we need to multiply the top & the bottom of the ~~eqs~~ by the conjugate ($2 + \sqrt{x}$)

$$\begin{aligned} &= \lim_{x \rightarrow 4} \frac{(2 - \sqrt{x})}{2\sqrt{x}(x-4)} \times \frac{(2 + \sqrt{x})}{(2 + \sqrt{x})} = \lim_{x \rightarrow 4} \frac{4 - x}{2\sqrt{x}(x-4)(2 + \sqrt{x})} \\ &= \lim_{x \rightarrow 4} \frac{-(x-4)}{2\sqrt{x}(x-4)(2 + \sqrt{x})} \\ &= \lim_{x \rightarrow 4} \frac{-1}{2\sqrt{x}(2 + \sqrt{x})} = \frac{-1}{2\sqrt{4}(2 + \sqrt{4})} \\ &= \boxed{\frac{-1}{16}} \end{aligned}$$

Algebraic Limits:الغايات الجبرية

The limits of numbers & variables together known as algebra of limits.

Properties

الخواص

$$1 - \lim_{x \rightarrow a} C = C$$

"The limit of a constant is equal to the constant"

$$2 - \lim_{x \rightarrow a} x = a$$

"The limit of x as x approaches a is equal to a "

$$3 - \lim_{x \rightarrow a} [f(x) \pm g(x)] = \lim_{x \rightarrow a} f(x) \pm \lim_{x \rightarrow a} g(x)$$

"The limit of a sum is the sum of the limits."
(likewise for differences)

$$4 - \lim_{x \rightarrow a} [C f(x)] = C \lim_{x \rightarrow a} f(x)$$

"The limit of a constant times a fun is the constant times the limit of the fun."

$$5 - \lim_{x \rightarrow a} [f(x) g(x)] = \lim_{x \rightarrow a} f(x) \lim_{x \rightarrow a} g(x)$$

"The limit of a product is the product of the limits"

$$6 - \lim_{x \rightarrow a} \frac{f(x)}{g(x)} = \frac{\lim_{x \rightarrow a} f(x)}{\lim_{x \rightarrow a} g(x)}, \quad \lim_{x \rightarrow a} g(x) \neq 0$$

"The limit of a quotient is the quotient of the limits, the denominator $\neq 0$."

$$7 - \lim_{x \rightarrow a} [f(x)]^n = \left(\lim_{x \rightarrow a} f(x) \right)^n, \quad n = \text{rational number}$$

مثال 4-12 or (-2-6), 8-56

"The limit of a power is the power of the limit, $n = \text{rational number}$."



$$B - \lim_{x \rightarrow a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x \rightarrow a} f(x)} \quad , \text{ "if the root on the right side exists"}$$

"The limit of a root is the root of the limit provided that the root exists"

Ex(7) Evaluate $\lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x}$

Solution

The direct substitution on x by 1 is not suitable idea as it gives $\frac{0}{0}$ = undetermined qu.

$$\begin{aligned} \therefore \lim_{x \rightarrow 1} \frac{x^2 + x - 2}{x^2 - x} &= \lim_{x \rightarrow 1} \frac{(x-1)(x+2)}{x(x-1)} \\ &= \lim_{x \rightarrow 1} \frac{x+2}{x} = \frac{1+2}{1} = \boxed{3} \end{aligned}$$

Trigonometric limits :- الدوال المثلثيةTheorem ①

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad (\theta \text{ in radians})$$

Theorem ② Sandwich Theorem

IF $g(x) \leq f(x) \leq h(x)$ for all x in open interval containing c , then,

$$\lim_{x \rightarrow c} g(x) = \lim_{x \rightarrow c} h(x) = L$$

Then, $\lim_{x \rightarrow c} f(x) = L$

Ex ③

IF $1 - \frac{x^2}{4} \leq u(x) \leq 1 + \frac{x^2}{2}$ for all $x \neq 0$
Find $\lim_{x \rightarrow 0} u(x)$, no matter how complicated u is

Solution

Since, $\lim_{x \rightarrow 0} 1 - \frac{x^2}{4} = \boxed{1}$

And, $\lim_{x \rightarrow 0} 1 + \frac{x^2}{2} = \boxed{1}$

The sandwich Theorem implies that $\lim_{x \rightarrow 0} u(x) = \boxed{1}$

Ex ④

IF $-|\theta| \leq \sin \theta \leq |\theta|$ For all θ , Find

$$\lim_{\theta \rightarrow 0} \sin \theta$$

Since, $\lim_{\theta \rightarrow 0} (-|\theta|) = \lim_{\theta \rightarrow 0} (|\theta|) = 0$



$$\therefore \lim_{\theta \rightarrow 0} \sin \theta = \boxed{0}$$

EX(10)

IF $0 \leq 1 - \cos \theta \leq |\theta|$ For all θ , Find $\lim_{\theta \rightarrow 0} (1 - \cos \theta)$

Sol-

Since, $\lim_{\theta \rightarrow 0} (0) = \lim_{\theta \rightarrow 0} |\theta| = 0$

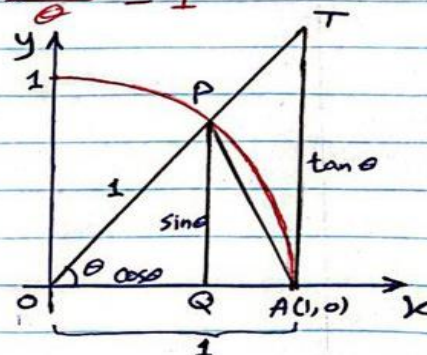
Then, $\lim_{\theta \rightarrow 0} (1 - \cos \theta) = 0 \Rightarrow \lim_{\theta \rightarrow 0} 1 - \lim_{\theta \rightarrow 0} \cos \theta = 0$

$\therefore 1 - \lim_{\theta \rightarrow 0} \cos \theta = 0 \Rightarrow \lim_{\theta \rightarrow 0} \cos \theta = \boxed{1}$

EX(11) proof that $\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1$

Sol-

The plan is to show that the right-hand & the left hand limits are both 1. Then we will know that the two-sided limit is 1 as well.



$$\text{Area } \triangle OAP < \text{area sector } OAP < \text{area } \triangle OAT$$

We can express these areas in terms of θ as follows

- $\text{Area } \triangle OAP = \frac{1}{2} \text{ base } \times \text{ height} = \frac{1}{2} (1)(\sin \theta) = \boxed{\frac{1}{2} \sin \theta}$
- $\text{Area sector } OAP = \frac{1}{2} r^2 \theta = \frac{1}{2} (1)^2 \theta = \boxed{\frac{\theta}{2}}$
- $\text{Area } \triangle OAT = \frac{1}{2} \text{ base } \times \text{ height} = \frac{1}{2} (1)(\tan \theta) = \boxed{\frac{1}{2} \tan \theta}$

Thus,

$$\left(\frac{1}{2} \sin \theta < \frac{\theta}{2} < \frac{1}{2} \tan \theta \right) \times \frac{2}{\sin \theta}$$



$$1 < \frac{\theta}{\sin \theta} < \frac{1}{\cos \theta}$$

take reverse,

$$(1 < \frac{\sin \theta}{\theta} < \cos \theta) \quad \lim_{\theta \rightarrow 0}$$

$$\lim_{\theta \rightarrow 0} 1 < \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} < \lim_{\theta \rightarrow 0} \cos \theta ;$$

$$1 < \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} < 1$$

From the Sandwich Theorem,

$$\boxed{\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1} \quad \text{O.K.}$$

Ex (12)

show that $\lim_{h \rightarrow 0} \frac{\cosh - 1}{h} = 0$

Sol.

Using the half-angle Formula

$$\cosh = 1 - 2 \sin^2\left(\frac{h}{2}\right)$$

$$\therefore \lim_{h \rightarrow 0} \frac{\cosh - 1}{h} = \lim_{h \rightarrow 0} \frac{1 - 2 \sin^2\left(\frac{h}{2}\right) - 1}{h}$$

$$= \lim_{h \rightarrow 0} \frac{-2 \sin^2\left(\frac{h}{2}\right)}{h}$$

put $\boxed{h = 2\theta} \Rightarrow h \rightarrow 0 = \theta \rightarrow 0$

$$= - \lim_{\theta \rightarrow 0} \frac{2 \sin \theta \times \sin \theta}{2\theta}$$

$$= - \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \times \sin \theta = - \lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} \times$$

$$\lim_{\theta \rightarrow 0} \sin \theta$$

$$= -(1)(0) = \boxed{0}$$

O.K.



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EX (13) | Show that $\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} = \frac{2}{5}$

Sol.

$$\lim_{x \rightarrow 0} \frac{\sin 2x}{5x} \times \frac{2/5}{2/5} = \lim_{x \rightarrow 0} \frac{(2/5) \times \sin 2x}{(2/5) \times 5x}$$

$$= \lim_{x \rightarrow 0} \frac{2}{5} \times \frac{\sin 2x}{2x}$$

$$= \frac{2}{5} \lim_{x \rightarrow 0} \frac{\sin 2x}{2x} \quad , \text{ put } 2x = 0$$

$$= \frac{2}{5} (1) = \boxed{\frac{2}{5}} \quad \underline{\underline{0 < x < \infty}}$$



Infinity as limits غالب/نهايات لا محدودة

Here, we investigate the behavior of a fun when the magnitude of the independent variables increasingly large, or $x \rightarrow \pm\infty$

will take examples to deal with all the cases associated with limit at Infinity

- Ex (14) Find $\lim_{x \rightarrow \infty} x^2$

Sol.

$$\lim_{x \rightarrow \infty} x^2 = (\infty)^2 = \boxed{+\infty}$$

- Ex (15) Find $\lim_{x \rightarrow -\infty} x^2$

Sol.

$$\lim_{x \rightarrow -\infty} x^2 = (-\infty)^2 = \boxed{+\infty}$$

- Ex (16) Find $\lim_{x \rightarrow \infty} x^3$

Sol.

$$\lim_{x \rightarrow \infty} x^3 = (\infty)^3 = \boxed{+\infty}$$

- Ex (17) Find $\lim_{x \rightarrow -\infty} x^3$

Sol.

$$\lim_{x \rightarrow -\infty} x^3 = (-\infty)^3 = \boxed{-\infty}$$



EX (8) Find $\lim_{x \rightarrow -\infty} (5 + x - x^3)$

Sol.

In such problems, we can ignore the small or insignificant parts of the polynomial equation & keep the heavy parts only.

هنا، يمكننا تجاهل الأجزاء الصغيرة والبقاء على الأجزاء الثقيلة فقط. (Equivalent)

$$\lim_{x \rightarrow -\infty} (5 + x - x^3) \approx \lim_{x \rightarrow -\infty} (-x^3) = -(-\infty)^3$$

$$= -(-\infty) = \boxed{+\infty}$$

EX (9) Find $\lim_{x \rightarrow -\infty} (3x^3 - 5x^4)$

Sol.

$$\lim_{x \rightarrow -\infty} (3x^3 - 5x^4) \approx \lim_{x \rightarrow -\infty} (-5x^4) = -5(-\infty)^4$$
$$= -5(+\infty)$$
$$= \boxed{-\infty}$$

EX (20) Find $\lim_{x \rightarrow \infty} \left(\frac{1}{x}\right)$

Sol.

$$\lim_{x \rightarrow \infty} \left(\frac{1}{x}\right) = \frac{1}{\infty} = \boxed{0}$$

"rational function"

Notes

$$\frac{1}{10} = 0.1$$

$$\frac{1}{100} = 0.01$$

$$\frac{1}{1000} = 0.001$$

$$\vdots$$

$$\frac{1}{\infty} = 0.00000000 \dots$$

$$= 0$$

EX (21) Find $\lim_{x \rightarrow -\infty} \frac{1}{x^2}$

Sol.

Similar to the EX (20)

$$\lim_{x \rightarrow -\infty} \frac{1}{x^2} = \frac{1}{(-\infty)^2} = \frac{1}{\infty} = \boxed{0}$$



Ex (22) Find $\lim_{x \rightarrow \infty} \frac{5x+2}{7x-x^2}$

Sol.

In such problems or rational fns, we look to the powers (degree)

For the numerator $\rightarrow 5x^{\textcircled{1}}+2$
For the denominator $\rightarrow 7x-x^{\textcircled{2}}$ } we have a bottom heavy fn

Note) Any rational fn with bottom heavy of the limit is going to (Fast), then its going to be zero

ملاحظة: في حالة الدوال الكسرية، ننظر إلى درجة مقامها البسط والمقام.
عندما كانت درجة أو أكثر من المقام أعلى من البسط، وكانت
الحدود متجهة إلى $(\pm\infty)$ ، فحينها الجواب يكون صفرًا.
وإذا كانت متساوية،

$$\lim_{x \rightarrow \infty} \frac{(5x+2)}{(7x-x^2)} \times \left(\frac{1/x^2}{1/x^2} \right) = \lim_{x \rightarrow \infty} \frac{5/x + 2/x^2}{7/x - 1}$$

$$= \frac{5/\infty + 2/\infty^2}{7/\infty - 1} = \frac{0+0}{0-1} = \frac{0}{-1} = \boxed{0}$$

Ex (23) Find $\lim_{x \rightarrow \infty} \frac{6x^2-4x}{3x^2+5}$

Sol.

Here the degrees are same, so the answer will be simply just divide the coefficients

$$\frac{6}{3} = \boxed{2}$$

ملاحظة: في حالة تساوي درجة البسط والمقام، فالجواب يكون ببساطة معامل البسط مقسوم على المعامل المقام.
في هذه الحالة، كلاهما x^2 ، لذا الجواب هو $6/3 = 2$.

$$\lim_{x \rightarrow \infty} \frac{(6x^2-4x)}{(3x^2+5)} \times \left(\frac{1/x^2}{1/x^2} \right) = \lim_{x \rightarrow \infty} \frac{6 - 4/x}{3 + 5/x^2} = \frac{6-0}{3-0}$$

$$\frac{6}{3} = \boxed{2}$$



Ex (24) Find $\lim_{x \rightarrow \infty} \frac{5x + 6x^2}{3x - 8}$

Sol-

$$\begin{aligned} \lim_{x \rightarrow \infty} \frac{(5x + 6x^2)}{(3x - 8)} \times \frac{1/x}{1/x} &= \lim_{x \rightarrow \infty} \frac{5 + 6x}{3 - 8/x} \\ &= \frac{5 + 6(\infty)}{3 - \frac{8}{\infty}} = \frac{5 + \infty}{3 - 0} = \boxed{+\infty} \end{aligned}$$

Ex (25) Find $\lim_{x \rightarrow -\infty} \frac{5 + 2x - 3x^3}{4x^2 + 9x - 7}$

Sol-

كل مربع يمكننا آخال اكدره الصغرة ونيفيا اكدره ذات الاكبر اقلية
بقدره كذا في الاكبر

$$\begin{aligned} \lim_{x \rightarrow -\infty} \frac{5 + 2x - 3x^3}{4x^2 + 9x - 7} &\sim \lim_{x \rightarrow -\infty} \frac{-3x^3}{4x^2} \\ &= \lim_{x \rightarrow -\infty} \frac{-3x}{4} = \frac{-3(\infty)}{4} \\ &= -\frac{3}{4}(\infty) = \boxed{-\infty} \end{aligned}$$

HW #4

① proof that $\lim_{t \rightarrow 0} \frac{\tan t \sec 2t}{3t} = \frac{1}{3}$

② Find $\lim_{x \rightarrow -\infty} \frac{5x - 7x^3}{2x^2 + 14x^3 - 9}$