5- Hyperbolic Functions الدوال الزائدية

- Functions which are associated with the geometry of a hyperbola are called hyperbolic Fns.
- Applications of hyperbolic Fns are:
  1. Transmission line theory نظرية خطوط النقل
  2. Catenary lines اسلاك القطارات الكهربائية

Hyperbolic Fns are:

① Hyperbolic sine of x ⟹ Pronounced as "shine x"
تلفظ بـ "شاين آكس"

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

② Hyperbolic Cosine of x ⟹ pron. as "Kosh x"
تلفظ بـ "كوش آكس"

$$\cosh x = \frac{e^x + e^{-x}}{2}$$

③ Hyperbolic tangent of x ⟹ pron. as "than x"
تلفظ بـ "ثان آكس"

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

④ Hyperbolic cosecant of x ⟹ pron. as "coshec x"
تلفظ بـ "كوشك آكس"

$$\operatorname{cosech} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

⑤ Hyperbolic secant of x ⟹ pron. as "shec x"
تلفظ بـ (شك آكس)

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

⑥ Hyperbolic cotangent of x ⟹ pron. as "koth x"
تلفظ بـ (كوث آكس)

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}}$$

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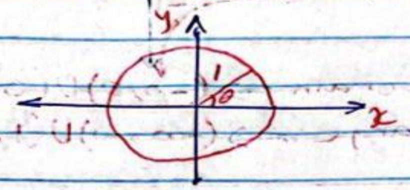
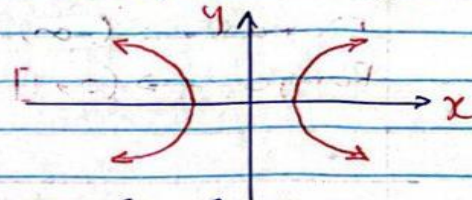
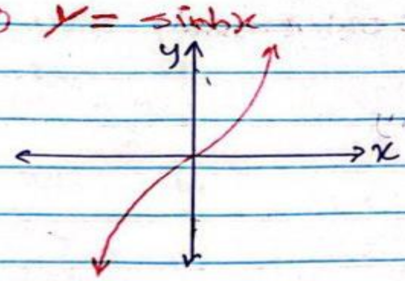
Trigonometric Fn's Vs hyperbolic Fun's :-

الدوال الدائرية مقابل الدوال الزائدية

Trigonometric Fn's	Hyperbolic Fn's
① Based on the unit circle.	① based on the hyperbola.
	
② $x^2+y^2=1$	② $x^2-y^2=1$
③ $\cos^2 x+\sin^2 x=1$	③ $\cosh^2 x-\sinh^2 x=1$

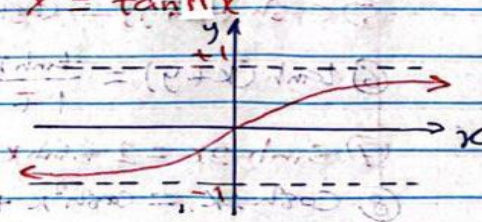
Graph of Hyperbolic Fn's :-

① $y=\sinh x$



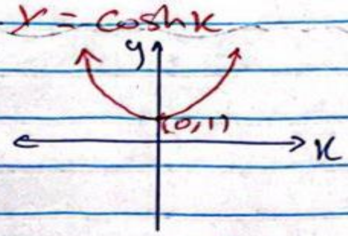
Domain → $(-\infty,+\infty)$
Range → $(-\infty,+\infty)$

③ $y=\tanh x$



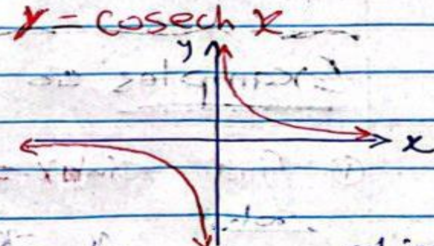
Domain → $(-\infty,+\infty)$
Range → $(-1,1)$

② $y=\cosh x$



Domain → $(-\infty,+\infty)$
Range → $[1,+\infty)$

④ $y=\operatorname{cosech} x$

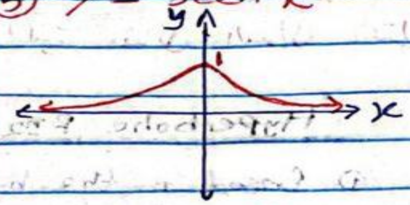


not included

Domain → $(-\infty,0)\cup(0,+\infty)$
Range → $(-\infty,0)\cup(0,+\infty)$

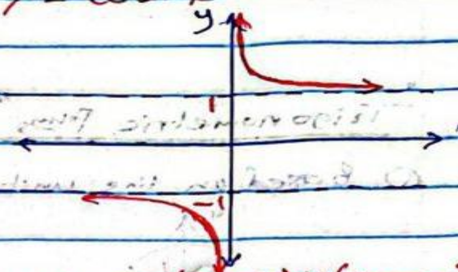
not included

(5) $y = \operatorname{sech} x$



Domain $\rightarrow (-\infty, +\infty)$
Range $\Rightarrow (0, 1]$

(6) $y = \coth x$



Domain $\rightarrow (-\infty, 0) \cup (0, \infty)$
Range $\Rightarrow (-\infty, -1) \cup (1, \infty)$

Hyperbolic F-ns Identities :-

1. $\cosh^2 x - \sinh^2 x = 1$
2. $1 - \tanh^2 x = \operatorname{sech}^2 x$
3. $\coth^2 x - 1 = \operatorname{cosech}^2 x$
4. $\sinh(x \mp y) = \sinh x \cosh y \mp \cosh x \sinh y$
5. $\cosh(x \mp y) = \cosh x \cosh y \mp \sinh x \sinh y$
6. $\tanh(x \mp y) = \dfrac{\tanh x \mp \tanh y}{1 \mp \tanh x \tanh y}$
7. $\sinh 2x = 2 \sinh x \cosh x$
8. $\cosh 2x = \cosh^2 x + \sinh^2 x$
   $= 2\cosh^2 x - 1$
   $= 1 + 2\sinh^2 x$
9. $\tanh 2x = \dfrac{2\tanh x}{1 + \tanh^2 x}$

Examples :-

(1) Find $\sinh x = 3$?

Sol.

$\sinh x = \dfrac{e^x - e^{-x}}{2} = 3 \rightarrow e^x - e^{-x} = 6$

$\rightarrow (e^x - e^{-x} - 6 = 0) * e^x$

$(e^x)^2 - e^{-x} e^{x} - 6e^x = 0 \Rightarrow (e^x)^2 - e^{(-x+x)} - 6e^x = 0$

$(e^{-x+x} = e^0 = 1)$

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$\therefore (e^x)^2 - 6e^x - 1 = 0$

$\therefore e^x = \frac{-(-6) \mp \sqrt{(-6)^2 - 4(1)(-1)}}{2(1)} = \frac{6 \mp \sqrt{36+4}}{2} = \frac{6 \mp \sqrt{40}}{2}$

$\therefore e^x = \frac{6 \mp 6.3246}{2} \Longrightarrow \therefore \boxed{\begin{array}{l} e^x = 6.1623 \\ e^x = -0.1623 \end{array}}$ or

② $y = 40 \cosh \frac{x}{40}$; $x = 25$

[Sol.]

$y = 40 \cosh \frac{25}{40} = 40 \cosh(0.625) = \cancel{40}^{20}\left(\frac{e^{0.625} + e^{-0.625}}{\cancel{2}_1}\right)$

$= 20(1.8682 + 0.5353) = \boxed{48.07}$

③ $y = 40 \cosh \frac{x}{40}$; $y = 54.3$

[Sol.]

$54.3 = 40 \cosh \frac{x}{40} \Longrightarrow \cosh \frac{x}{40} = \frac{54.3}{40} = 1.3575$

$\frac{e^{x/40} + e^{-x/40}}{2} = 1.3575 \Longrightarrow e^{x/40} + e^{-x/40} = 2.715 \Longrightarrow$

$(e^{x/40} + e^{-x/40} - 2.715 = 0) \times e^{x/40}$

$(e^{x/40})^2 + 1 - 2.715\, e^{x/40} = 0 \Longrightarrow (e^{x/40})^2 - 2.715\, e^{x/40} + 1 = 0$

$\therefore e^{x/40} = \frac{-(-2.715) \mp \sqrt{(-2.715)^2 - 4(1)(1)}}{2(1)} = \frac{2.715 \mp \sqrt{3.3712}}{2}$

$\therefore \left[e^{x/40} = 2.2756 \quad \text{or} \quad 0.43945 \right] \times \ln$

$\frac{x}{40} = \ln(2.2756)$ or $\frac{x}{40} = \ln(0.43945)$

$\therefore \boxed{x = \mp 32.89}$

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6- Inverse Hyperbolic Functions : معكوس الدوال الزائدية

Inverse hyperbolic Fns are denoted by (يرمز لها) using the -1 notation.

If $y = \sinh x$, then $x = \sinh^{-1} y$, and
If $y = \operatorname{sech} x$, then $x = \operatorname{sech}^{-1} y$, and so on.

① $y = \sinh^{-1} x = \ln\left[x + \sqrt{x^2+1}\right]$, Domain $= [1, \infty)$

② $y = \cosh^{-1} x = \ln\left[x + \sqrt{x^2-1}\right]$, Domain $= (-\infty, \infty)$

③ $y = \tanh^{-1} x = \frac{1}{2} \ln\left|\frac{1+x}{1-x}\right|$, Domain $= (-1, 1)$

④ $y = \operatorname{cosech}^{-1} x = \ln\left|\frac{1}{x} + \frac{\sqrt{1+x^2}}{|x|}\right|$, Dom. $= (-\infty, 0) \cup (0, \infty)$

⑤ $y = \operatorname{sech}^{-1} x = \ln\left|\frac{1+\sqrt{1-x^2}}{x}\right|$, Domain $= (0, 1]$

⑥ $y = \coth^{-1} x = \frac{1}{2} \ln\left|\frac{x+1}{x-1}\right|$, Domain $= (-\infty, -1) \cup (1, \infty)$

Examples

① Evaluate $\sinh^{-1} 2$

Sol.

$$\sinh^{-1} 2 = \ln\left[2 + \sqrt{2^2+1}\right] = \ln\left[2 + \sqrt{5}\right]$$
$$= \ln(4.2361) = \boxed{1.4436}$$

② Evaluate $\cosh^{-1} 1.4$

Sol.

$$\cosh^{-1} 1.4 = \ln\left[1.4 + \sqrt{1.4^2 - 1}\right] = \ln\left[1.4 + 0.979\right]$$
$$= \ln(2.379) = \boxed{0.867}$$

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