8.3 Trigonometric Substitutions

So far we have seen that it sometimes helps to replace a subexpression of a function by a single variable. Occasionally it can help to replace the original variable by something more complicated. This seems like a "reverse" substitution, but it is really no different in principle than ordinary substitution.

EXAMPLE 8.3.1 Evaluate $\int \sqrt{1-x^2}\,dx$. Let $x = \sin u$ so $dx = \cos u\,du$. Then

$$\int \sqrt{1-x^2}\,dx = \int \sqrt{1-\sin^2 u}\cos u\,du = \int \sqrt{\cos^2 u}\cos u\,du.$$

We would like to replace $\sqrt{\cos^2 u}$ by $\cos u$, but this is valid only if $\cos u$ is positive, since $\sqrt{\cos^2 u}$ is positive. Consider again the substitution $x = \sin u$. We could just as well think of this as $u = \arcsin x$. If we do, then by the definition of the arcsine, $-\pi/2 \le u \le \pi/2$, so $\cos u \ge 0$. Then we continue:

$$\begin{aligned}\int \sqrt{\cos^2 u}\cos u\,du = \int \cos^2 u\,du = \int \frac{1+\cos 2u}{2}\,du = \frac{u}{2} + \frac{\sin 2u}{4} + C\\ = \frac{\arcsin x}{2} + \frac{\sin(2\arcsin x)}{4} + C.\end{aligned}$$

This is a perfectly good answer, though the term $\sin(2\arcsin x)$ is a bit unpleasant. It is possible to simplify this. Using the identity $\sin 2x = 2\sin x\cos x$, we can write $\sin 2u = 2\sin u\cos u = 2\sin(\arcsin x)\sqrt{1-\sin^2 u} = 2x\sqrt{1-\sin^2(\arcsin x)} = 2x\sqrt{1-x^2}$. Then the full antiderivative is

$$\frac{\arcsin x}{2} + \frac{2x\sqrt{1-x^2}}{4} = \frac{\arcsin x}{2} + \frac{x\sqrt{1-x^2}}{2} + C.$$

□

This type of substitution is usually indicated when the function you wish to integrate contains a polynomial expression that might allow you to use the fundamental identity $\sin^2 x + \cos^2 x = 1$ in one of three forms:

$$\cos^2 x = 1 - \sin^2 x \qquad \sec^2 x = 1 + \tan^2 x \qquad \tan^2 x = \sec^2 x - 1.$$

If your function contains $1-x^2$, as in the example above, try $x = \sin u$; if it contains $1+x^2$ try $x = \tan u$; and if it contains x^2-1, try $x = \sec u$. Sometimes you will need to try something a bit different to handle constants other than one.

EXAMPLE 8.3.2 Evaluate $\int \sqrt{4-9x^2}\,dx$. We start by rewriting this so that it looks more like the previous example:

$$\int \sqrt{4-9x^2}\,dx = \int \sqrt{4(1-(3x/2)^2)}\,dx = \int 2\sqrt{1-(3x/2)^2}\,dx.$$

Now let $3x/2 = \sin u$ so $(3/2)\,dx = \cos u\,du$ or $dx = (2/3)\cos u\,du$. Then

$$\begin{aligned}
\int 2\sqrt{1-(3x/2)^2}\,dx &= \int 2\sqrt{1-\sin^2 u}\,(2/3)\cos u\,du = \frac{4}{3}\int \cos^2 u\,du \\
&= \frac{4u}{6} + \frac{4\sin 2u}{12} + C \\
&= \frac{2\arcsin(3x/2)}{3} + \frac{2\sin u\cos u}{3} + C \\
&= \frac{2\arcsin(3x/2)}{3} + \frac{2\sin(\arcsin(3x/2))\cos(\arcsin(3x/2))}{3} + C \\
&= \frac{2\arcsin(3x/2)}{3} + \frac{2(3x/2)\sqrt{1-(3x/2)^2}}{3} + C \\
&= \frac{2\arcsin(3x/2)}{3} + \frac{x\sqrt{4-9x^2}}{2} + C,
\end{aligned}$$

using some of the work from example 8.3.1. □

EXAMPLE 8.3.3 Evaluate $\int \sqrt{1+x^2}\,dx$. Let $x = \tan u$, $dx = \sec^2 u\,du$, so

$$\int \sqrt{1+x^2}\,dx = \int \sqrt{1+\tan^2 u}\sec^2 u\,du = \int \sqrt{\sec^2 u}\sec^2 u\,du.$$

Since $u = \arctan(x)$, $-\pi/2 \le u \le \pi/2$ and $\sec u \ge 0$, so $\sqrt{\sec^2 u} = \sec u$. Then

$$\int \sqrt{\sec^2 u}\sec^2 u\,du = \int \sec^3 u\,du.$$

In problems of this type, two integrals come up frequently: $\int \sec^3 u\,du$ and $\int \sec u\,du$. Both have relatively nice expressions but they are a bit tricky to discover.

First we do $\int \sec u\, du$, which we will need to compute $\displaystyle\int \sec^3 u\, du$:

$$\begin{aligned}\int \sec u\, du &= \int \sec u \frac{\sec u + \tan u}{\sec u + \tan u}\, du\\ &= \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u}\, du.\end{aligned}$$

Now let $w = \sec u + \tan u$, $dw = \sec u \tan u + \sec^2 u\, du$, exactly the numerator of the function we are integrating. Thus

$$\begin{aligned}\int \sec u\, du = \int \frac{\sec^2 u + \sec u \tan u}{\sec u + \tan u}\, du = \int \frac{1}{w}\, dw &= \ln|w| + C\\ &= \ln|\sec u + \tan u| + C.\end{aligned}$$

Now for $\displaystyle\int \sec^3 u\, du$:

$$\begin{aligned}\sec^3 u &= \frac{\sec^3 u}{2} + \frac{\sec^3 u}{2} = \frac{\sec^3 u}{2} + \frac{(\tan^2 u + 1)\sec u}{2}\\ &= \frac{\sec^3 u}{2} + \frac{\sec u \tan^2 u}{2} + \frac{\sec u}{2} = \frac{\sec^3 u + \sec u \tan^2 u}{2} + \frac{\sec u}{2}.\end{aligned}$$

We already know how to integrate $\sec u$, so we just need the first quotient. This is "simply" a matter of recognizing the product rule in action:

$$\int \sec^3 u + \sec u \tan^2 u\, du = \sec u \tan u.$$

So putting these together we get

$$\int \sec^3 u\, du = \frac{\sec u \tan u}{2} + \frac{\ln|\sec u + \tan u|}{2} + C,$$

and reverting to the original variable x:

$$\begin{aligned}\int \sqrt{1+x^2}\, dx &= \frac{\sec u \tan u}{2} + \frac{\ln|\sec u + \tan u|}{2} + C\\ &= \frac{\sec(\arctan x)\tan(\arctan x)}{2} + \frac{\ln|\sec(\arctan x) + \tan(\arctan x)|}{2} + C\\ &= \frac{x\sqrt{1+x^2}}{2} + \frac{\ln|\sqrt{1+x^2} + x|}{2} + C,\end{aligned}$$

using $\tan(\arctan x) = x$ and $\sec(\arctan x) = \sqrt{1 + \tan^2(\arctan x)} = \sqrt{1+x^2}$. □

Exercises 8.3.

Find the antiderivatives.

1. $\int \csc x\,dx \Rightarrow$

2. $\int \csc^3 x\,dx \Rightarrow$

3. $\int \sqrt{x^2-1}\,dx \Rightarrow$

4. $\int \sqrt{9+4x^2}\,dx \Rightarrow$

5. $\int x\sqrt{1-x^2}\,dx \Rightarrow$

6. $\int x^2\sqrt{1-x^2}\,dx \Rightarrow$

7. $\int \frac{1}{\sqrt{1+x^2}}\,dx \Rightarrow$

8. $\int \sqrt{x^2+2x}\,dx \Rightarrow$

9. $\int \frac{1}{x^2(1+x^2)}\,dx \Rightarrow$

10. $\int \frac{x^2}{\sqrt{4-x^2}}\,dx \Rightarrow$

11. $\int \frac{\sqrt{x}}{\sqrt{1-x}}\,dx \Rightarrow$

12. $\int \frac{x^3}{\sqrt{4x^2-1}}\,dx \Rightarrow$

13. Compute $\int \sqrt{x^2+1}\,dx$. (Hint: make the substitution $x = \sinh(u)$ and then use exercise 6 in section 4.11.)

14. Fix $t > 0$. The shaded region in the left-hand graph in figure 4.11.2 is bounded by $y = x\tanh t$, $y = 0$, and $x^2 - y^2 = 1$. Prove that twice the area of this region is t, as claimed in section 4.11.

8.4 Integration by Parts

We have already seen that recognizing the product rule can be useful, when we noticed that

$$\int \sec^3 u + \sec u \tan^2 u\,du = \sec u \tan u.$$

As with substitution, we do not have to rely on insight or cleverness to discover such antiderivatives; there is a technique that will often help to uncover the product rule.

Start with the product rule:

$$\frac{d}{dx} f(x)g(x) = f'(x)g(x) + f(x)g'(x).$$

We can rewrite this as

$$f(x)g(x) = \int f'(x)g(x)\,dx + \int f(x)g'(x)\,dx,$$

and then

$$\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx.$$

This may not seem particularly useful at first glance, but it turns out that in many cases we have an integral of the form

$$\int f(x)g'(x)\,dx$$

but that

$$\int f'(x)g(x)\,dx$$

is easier. This technique for turning one integral into another is called **integration by parts**, and is usually written in more compact form. If we let $u = f(x)$ and $v = g(x)$ then $du = f'(x)\,dx$ and $dv = g'(x)\,dx$ and

$$\int u\,dv = uv - \int v\,du.$$

To use this technique we need to identify likely candidates for $u = f(x)$ and $dv = g'(x)\,dx$.

EXAMPLE 8.4.1 Evaluate $\int x \ln x\,dx$. Let $u = \ln x$ so $du = 1/x\,dx$. Then we must let $dv = x\,dx$ so $v = x^2/2$ and

$$\int x \ln x\,dx = \frac{x^2 \ln x}{2} - \int \frac{x^2}{2}\frac{1}{x}\,dx = \frac{x^2 \ln x}{2} - \int \frac{x}{2}\,dx = \frac{x^2 \ln x}{2} - \frac{x^2}{4} + C.$$

□

EXAMPLE 8.4.2 Evaluate $\int x \sin x\,dx$. Let $u = x$ so $du = dx$. Then we must let $dv = \sin x\,dx$ so $v = -\cos x$ and

$$\int x \sin x\,dx = -x\cos x - \int -\cos x\,dx = -x\cos x + \int \cos x\,dx = -x\cos x + \sin x + C.$$

□

EXAMPLE 8.4.3 Evaluate $\int \sec^3 x\,dx$. Of course we already know the answer to this, but we needed to be clever to discover it. Here we'll use the new technique to discover the antiderivative. Let $u = \sec x$ and $dv = \sec^2 x\,dx$. Then $du = \sec x \tan x\,dx$ and $v = \tan x$

and

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x\tan x - \int \tan^2 x\sec x\,dx\\ &= \sec x\tan x - \int (\sec^2 x - 1)\sec x\,dx\\ &= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx.\end{aligned}$$

At first this looks useless—we're right back to $\int \sec^3 x\,dx$. But looking more closely:

$$\begin{aligned}\int \sec^3 x\,dx &= \sec x\tan x - \int \sec^3 x\,dx + \int \sec x\,dx\\ \int \sec^3 x\,dx + \int \sec^3 x\,dx &= \sec x\tan x + \int \sec x\,dx\\ 2\int \sec^3 x\,dx &= \sec x\tan x + \int \sec x\,dx\\ \int \sec^3 x\,dx &= \frac{\sec x\tan x}{2} + \frac{1}{2}\int \sec x\,dx\\ &= \frac{\sec x\tan x}{2} + \frac{\ln|\sec x + \tan x|}{2} + C.\end{aligned}$$

□

EXAMPLE 8.4.4 Evaluate $\int x^2\sin x\,dx$. Let $u = x^2$, $dv = \sin x\,dx$; then $du = 2x\,dx$ and $v = -\cos x$. Now $\int x^2\sin x\,dx = -x^2\cos x + \int 2x\cos x\,dx$. This is better than the original integral, but we need to do integration by parts again. Let $u = 2x$, $dv = \cos x\,dx$; then $du = 2$ and $v = \sin x$, and

$$\begin{aligned}\int x^2\sin x\,dx &= -x^2\cos x + \int 2x\cos x\,dx\\ &= -x^2\cos x + 2x\sin x - \int 2\sin x\,dx\\ &= -x^2\cos x + 2x\sin x + 2\cos x + C.\end{aligned}$$

□

Such repeated use of integration by parts is fairly common, but it can be a bit tedious to accomplish, and it is easy to make errors, especially sign errors involving the subtraction in the formula. There is a nice tabular method to accomplish the calculation that minimizes the chance for error and speeds up the whole process. We illustrate with the previous example. Here is the table:

sign	u	dv
	x^2	$\sin x$
$-$	$2x$	$-\cos x$
	2	$-\sin x$
$-$	0	$\cos x$

or

u	dv
x^2	$\sin x$
$-2x$	$-\cos x$
2	$-\sin x$
0	$\cos x$

To form the first table, we start with u at the top of the second column and repeatedly compute the derivative; starting with dv at the top of the third column, we repeatedly compute the antiderivative. In the first column, we place a "$-$" in every second row. To form the second table we combine the first and second columns by ignoring the boundary; if you do this by hand, you may simply start with two columns and add a "$-$" to every second row.

To compute with this second table we begin at the top. Multiply the first entry in column u by the second entry in column dv to get $-x^2 \cos x$, and add this to the integral of the product of the second entry in column u and second entry in column dv. This gives:

$$-x^2 \cos x + \int 2x \cos x \, dx,$$

or exactly the result of the first application of integration by parts. Since this integral is not yet easy, we return to the table. Now we multiply twice on the diagonal, $(x^2)(-\cos x)$ and $(-2x)(-\sin x)$ and then once straight across, $(2)(-\sin x)$, and combine these as

$$-x^2 \cos x + 2x \sin x - \int 2 \sin x \, dx,$$

giving the same result as the second application of integration by parts. While this integral is easy, we may return yet once more to the table. Now multiply three times on the diagonal to get $(x^2)(-\cos x)$, $(-2x)(-\sin x)$, and $(2)(\cos x)$, and once straight across, $(0)(\cos x)$. We combine these as before to get

$$-x^2 \cos x + 2x \sin x + 2 \cos x + \int 0 \, dx = -x^2 \cos x + 2x \sin x + 2 \cos x + C.$$

Typically we would fill in the table one line at a time, until the "straight across" multiplication gives an easy integral. If we can see that the u column will eventually become zero, we can instead fill in the whole table; computing the products as indicated will then give the entire integral, including the "$+C$", as above.

Exercises 8.4.

Find the antiderivatives.

1. $\int x\cos x\,dx \Rightarrow$

2. $\int x^2\cos x\,dx \Rightarrow$

3. $\int xe^x\,dx \Rightarrow$

4. $\int xe^{x^2}\,dx \Rightarrow$

5. $\int \sin^2 x\,dx \Rightarrow$

6. $\int \ln x\,dx \Rightarrow$

7. $\int x\arctan x\,dx \Rightarrow$

8. $\int x^3\sin x\,dx \Rightarrow$

9. $\int x^3\cos x\,dx \Rightarrow$

10. $\int x\sin^2 x\,dx \Rightarrow$

11. $\int x\sin x\cos x\,dx \Rightarrow$

12. $\int \arctan(\sqrt{x})\,dx \Rightarrow$

13. $\int \sin(\sqrt{x})\,dx \Rightarrow$

14. $\int \sec^2 x\csc^2 x\,dx \Rightarrow$