1-5- Limits and continuity :

Limits : The limit of $F(t)$ as t approaches C is the number L if :
Given any radius $\varepsilon > 0$ about L there exists a radius $\delta > 0$ about C such that for all t, $0 < |t - C| < \delta$ implies $|F(t) - L| < \varepsilon$ and we can write it as :

$$\lim_{t \to C} F(t) = L$$

The limit of a function $F(t)$ as $t \to C$ never depend on what happens when $t = C$.

Right hand limit : $\lim_{t \to C^+} F(t) = L$

The limit of the function $F(t)$ as $t \to C$ from the right equals L if :
Given any $\varepsilon > 0$ (radius about L) there exists a $\delta > 0$ (radius to the right of C) such that for all t :

$$C < t < C + \delta \Rightarrow |F(t) - L| < \varepsilon$$

Left hand limit : $\lim_{t \to C^-} F(t) = L$

The limit of the function $F(t)$ as $t \to C$ from the left equal L if :
Given any $\varepsilon > 0$ there exists a $\delta > 0$ such that for all t :

$$C - \delta < t < C \Rightarrow |F(t) - L| < \varepsilon$$

Note that – A function $F(t)$ has a limit at point C if and only if the right hand and the left hand limits at C exist and equal . In symbols :

$$\lim_{t \to C} F(t) = L \Leftrightarrow \lim_{t \to C^+} F(t) = L \quad and \quad \lim_{t \to C^-} F(t) = L$$

The limit combinations theorems :

1) $\lim[F_1(t) \mp F_2(t)] = \lim F_1(t) \mp \lim F_2(t)$
2) $\lim[F_1(t) * F_2(t)] = \lim F_1(t) * \lim F_2(t)$
3) $\lim \dfrac{F_1(t)}{F_2(t)} = \dfrac{\lim F_1(t)}{\lim F_2(t)}$ where $\lim F_2(t) \neq 0$
4) $\lim[k * F_1(t)] = k * \lim F_1(t) \quad \forall k$
5) $\lim_{\theta \to 0} \dfrac{Sin\theta}{\theta} = 1$

provided that θ is measured in radius

The limits (in 1 – 4) are all to be taken as $t \to C$ and $F_1(t)$ and $F_2(t)$ are to be real functions .

Thm. -1 : The sandwich theorem : Suppose that $f(t) \le g(t) \le h(t)$ for all $t \neq C$ in some interval about C and that $f(t)$ and $h(t)$ approaches the same limit L as $t \to C$, then :

$$\lim_{t \to C} g(t) = L$$

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Infinity as a limit :

1.The limit of the function $f(x)$ as x approaches infinity is the number L: $\lim_{x\to\infty} f(x)=L$ **. If , given any $\varepsilon>0$ there exists a number M such that for all x : $M<x \Rightarrow |f(x)-L|<\varepsilon$.**

2. The limit of $f(x)$ as x approaches negative infinity is the number L : $\lim_{x\to-\infty} f(x)=L$ **. If , given any $\varepsilon>0$ there exists a number N such that for all x : $x<N \Rightarrow |f(x)-L|<\varepsilon$.**

The following facts are some times abbreviated by saying :

- a) As x approaches 0 from the right , $1/x$ tends to ∞ .
- b) As x approaches 0 from the left , $1/x$ tends to $-\infty$.
- c) As x tends to ∞ , $1/x$ approaches 0 .
- d) As x tends to $-\infty$, $1/x$ approaches 0 .

Continuity :

Continuity at an interior point : **A function $y=f(x)$ is continuous at an interior point C of its domain if :** $\lim_{x\to C} f(x)=f(C)$.

Continuity at an endpoint : **A function $y=f(x)$ is continuous at a left endpoint a of its domain if :** $\lim_{x\to a^+} f(x)=f(a)$.

A function $y=f(x)$ is continuous at a right endpoint b of its domain if: $\lim_{t\to b^-} f(x)=f(b)$.

Continuous function : **A function is continuous if it is continuous at each point of its domain .**

Discontinuity at a point : **If a function f is not continuous at a point C , we say that f is discontinuous at C , and call C a point of discontinuity of f .**

The continuity test : **The function $y=f(x)$ is continuous at $x=C$ if and only if all three of the following statements are true :**

1) $f(C)$ exist (C is in the domain of f) .
2) $\lim_{x\to C} f(x)$ exists (f has a limit as $x\to C$) .
3) $\lim_{x\to C} f(x)=f(C)$ (the limit equals the function value) .

Thm.-2 : The limit combination theorem for continuous function :

If the function f and g are continuous at $x=C$, then all of the following combinations are continuous at $x=C$:

$$1)\ f\mp g \quad 2)\ f.g \quad 3)\ k.g\ \ \forall k \quad 4)\ g_o f\ ,\ f_o g \quad 5)\ f/g$$

provided $g(C)\neq 0$

Thm.-3 : **A function is continuous at every point at which it has a derivative . That is , if $y=f(x)$ has a derivative $f'(C)$ at $x=C$, then f is continuous at $x=C$.**

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EX-12 – **Find :**

1) $\lim\limits_{x\to 0}\dfrac{5x^3+8x^2}{3x^4-16x^2}$, 2) $\lim\limits_{x\to a}\dfrac{x^3-a^3}{x^4-a^4}$

3) $\lim\limits_{x\to 0}\dfrac{Sin5x}{Sin3x}$, 4) $\lim\limits_{y\to 0}\dfrac{tan2y}{3y}$

5) $\lim\limits_{x\to 0}\dfrac{Sin2x}{2x^2+x}$, 6) $\lim\limits_{x\to\infty}\left(1+Cos\dfrac{1}{x}\right)$

7) $\lim\limits_{x\to\infty}\dfrac{3x^3+5x^2-7}{10x^3-11x+5}$, 8) $\lim\limits_{y\to\infty}\dfrac{3y+7}{y^2-2}$

9) $\lim\limits_{x\to\infty}\dfrac{x^3-1}{2x^2-7x+5}$, 10) $\lim\limits_{x\to -1^-}\dfrac{1}{x+1}$

11) $\lim\limits_{x\to 0}Cos\left(1-\dfrac{Sinx}{x}\right)$, 12) $\lim\limits_{x\to 0}Sin\left(\dfrac{\pi}{2}Cos(tan\,x)\right)$

S0l.-

1) $\lim\limits_{x\to 0}\dfrac{5x^3+8x^2}{3x^4-16x^2}=\lim\limits_{x\to 0}\dfrac{5x+8}{3x^2-16}=\dfrac{0+8}{0-16}=-\dfrac{1}{2}$

2) $\lim\limits_{x\to a}\dfrac{x^3-a^3}{x^4-a^4}=\lim\limits_{x\to a}\dfrac{(x-a)(x^2+ax+a^2)}{(x-a)(x+a)(x^2+a^2)}=\dfrac{a^2+a^2+a^2}{(a+a)(a^2+a^2)}=\dfrac{3}{4a}$

3) $\lim\limits_{x\to 0}\dfrac{5\dfrac{Sin5x}{5x}}{3\dfrac{Sin3x}{3x}}=\dfrac{5}{3}\cdot\dfrac{\lim\limits_{5x\to 0}\dfrac{Sin5x}{5x}}{\lim\limits_{3x\to 0}\dfrac{Sin3x}{3x}}=\dfrac{5}{3}$

4) $\lim\limits_{y\to 0}\dfrac{tan2y}{3y}=\dfrac{2}{3}\cdot\lim\limits_{2y\to 0}\dfrac{Sin2y}{2y}\cdot\lim\limits_{y\to 0}\dfrac{1}{Cos2y}=\dfrac{2}{3}$

5) $\lim\limits_{x\to 0}\dfrac{Sin2x}{2x^2+x}=2\lim\limits_{2x\to 0}\dfrac{Sin2x}{2x}\cdot\lim\limits_{x\to 0}\dfrac{1}{2x+1}=2$

6) $\lim\limits_{x\to\infty}\left(1+Cos\dfrac{1}{x}\right)=1+Cos0=2$

7) $\lim\limits_{x\to\infty}\dfrac{3x^3+5x^2-7}{10x^3-11x+5}=\lim\limits_{x\to\infty}\dfrac{3+\dfrac{5}{x}-\dfrac{7}{x^3}}{10-\dfrac{11}{x^2}+\dfrac{5}{x^3}}=\dfrac{3}{10}$

8) $\lim\limits_{y\to\infty}\dfrac{3y+7}{y^2-2}=\lim\limits_{y\to\infty}\dfrac{\dfrac{3}{y}+\dfrac{7}{y^2}}{1-\dfrac{2}{y^2}}=\dfrac{0}{1}=0$

9) $\lim\limits_{x\to\infty}\dfrac{x^3-1}{2x^2-7x+5}=\lim\limits_{x\to\infty}\dfrac{1-\dfrac{1}{x^3}}{\dfrac{2}{x}-\dfrac{7}{x^2}+\dfrac{5}{x^3}}=\dfrac{1}{0}=\infty$

10) $\lim\limits_{x\to -1^-}\dfrac{1}{x+1}=\dfrac{1}{-1+1}=-\infty$

11) $\lim\limits_{x\to 0}Cos\left(1-\dfrac{Sinx}{x}\right)=Cos\left(1-\lim\limits_{x\to 0}\dfrac{Sinx}{x}\right)=Cos0=1$

12) $\lim\limits_{x\to 0}Sin\left(\dfrac{\pi}{2}Cos(tan\,x)\right)=Sin\left(\dfrac{\pi}{2}Cos(tan0)\right)=Sin\left(\dfrac{\pi}{2}Cos0\right)=Sin\dfrac{\pi}{2}=1$

EX-13- Test continuity for the following function :

$$f(x)=\begin{cases} x^2-1 & -1\le x<0 \\ 2x & 0\le x<1 \\ 1 & x=1 \\ -2x+4 & 1<x\le 2 \\ 0 & 2<x\le 3 \end{cases}$$

Sol.- We test the continuity at midpoints $x = 0, 1, 2$ and endpoints $x = -1, 3$.

At $x=0 \Rightarrow$ $f(0)=2*0=0$

$$\lim_{x\to 0^-} f(x)=\lim_{x\to 0}(x^2-1)=-1$$

$$\lim_{x\to 0^+} f(x)=\lim_{x\to 0} 2x=0\ne \lim_{x\to 0^-} f(x)$$

Since $\lim_{x\to 0} f(x)$ doesn't exist

Hence the function discontinuous at $x=0$

At $x=1 \Rightarrow$ $f(1)=1$

$$\lim_{x\to 1^-} f(x)=\lim_{x\to 1} 2x=2$$

$$\lim_{x\to 1^+} f(x)=\lim_{x\to 1}(-2x+4)=2=\lim_{x\to 1^-} f(x)=\lim_{x\to 1} f(x)$$

Since $\lim_{x\to 1} f(x)\ne f(1)$

Hence the function is discontinuous at $x=1$

At $x=2 \Rightarrow$ $f(2)=-2*2+4=0$

$$\lim_{x\to 2^-} f(x)=\lim_{x\to 2}(-2x+4)=0$$

$$\lim_{x\to 2^+} f(x)=\lim_{x\to 2} 0=0=\lim_{x\to 2^-} f(x)=\lim_{x\to 2} f(x)$$

Since $\lim_{x\to 2} f(x)=f(2)=0$

Hence the function is continuous at $x=2$

At $x=-1 \Rightarrow$ $f(-1)=(-1)^2-1=0$

$$\lim_{x\to -1^+} f(x)=\lim_{x\to -1}(x^2-1)=0=f(-1)$$

Hence the function is continuous at $x=-1$

At $x=3 \Rightarrow$ $f(3)=0$

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3} 0=0=f(3)$$

Hence the function is continuous at $x=3$

EX-14- **What value should be assigned to a to make the function :**

$$f(x)=\begin{Bmatrix} x^2-1 & x<3 \\ 2ax & x\ge 3 \end{Bmatrix} \quad \text{continuous at } x=3\text{ ?}$$

Sol. –

$$\lim_{x\to 3^-} f(x)=\lim_{x\to 3^+} f(x) \Rightarrow \lim_{x\to 3}(x^2-1)=\lim_{x\to 3} 2ax \Rightarrow \quad 8=6a \Rightarrow \quad a=\frac{4}{3}$$

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Problems – 1

1. The steel in railroad track expands when heated . For the track temperature encountered in normal outdoor use , the length S of a piece of track is related to its temperature t by a linear equation . An experiment with a piece of track gave the following measurements :

$$t_1 = 65^o F \quad , \quad S_1 = 35 \ ft$$
$$t_2 = 135^o F \quad , \quad S_2 = 35.16 \ ft$$

Write a linear equation for the relation between S and t.
(ans.: S=0.0023t+34.85)

2. Three of the following four points lie on a circle center the origin . Which are they , and what is the radius of the circle ?
$A(-1.7)$, $B(5,-5)$, $C(-7,5)$ and $D(7,-1)$. *(ans.: A,B,D;√50)*

3. A and B are the points $(3,4)$ and $(7,1)$ respectively . Use Pythagoras theorem to prove that OA is perpendicular to AB . Calculate the slopes of OA and AB , and find their product . *(ans.: 4/3, -3/4;-1)*

4. $P(-2,-4)$, $Q(-5,-2)$, $R(2,1)$ and S are the vertices of a parallelogram . Find the coordinates of M , the point of intersection of the diagonals and of S.
(ans.: M(0,-3/2) , S(5,-1))

5. Calculate the area of the triangle formed by the line $3x-7y+4 =0$, and the axes . *(ans.: 8/21)*

6. Find the equation of the straight line through $P(7,5)$ perpendicular to the straight line AB whose equation is $3x + 4y -16 = 0$. Calculate the length of the perpendicular from P and AB. *(ans.: 3y-4x+13=0;5)*

7. $L(-1,0)$, $M(3,7)$ and $N(5,-2)$ are the mid-points of the sides BC , CA and AB respectively of the triangle ABC. Find the equation of AB. *(ans.:4y=7x-43)*

8. The straight line x – y – 6 = 0 cuts the curve $y^2 = 8x$ at P and Q . Calculate the length of PQ . *(ans.:16√2)*

9. A line is drawn through the point $(2,3)$ making an angle of 45^o with the positive direction of the x-axis and it meets the line $x = 6$ at P . Find the distance of P from the origin O , and the equation of the line through P perpendicular to OP. *(ans.: √85, 7y+6x-85=0)*

10. The vertices of a quadrilateral $ABCD$ are $A(4,0)$, $B(14,11)$, $C(0,6)$ and $D(-10,-5)$. Prove that the diagonals AC and BD bisect each other at right angles , and that the length of BD is four times that of AC .

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11. The coordinates of the vertices *A, B* and *C* of the triangle *ABC* are *(-3,7)*, *(2,19)* and *(10,7)* respectively :

a) Prove that the triangle is isosceles.

b) Calculate the length of the perpendicular from *B* to *AC*, and use it to find the area of the triangle . *(ans.:12,78)*

12. Find the equations of the lines which pass through the point of intersection of the lines $x - 3y = 4$ and $3x + y = 2$ and are respectively parallel and perpendicular to the line $3x + 4y = 0$.

(ans.:4y+3x+1=0;3y-4x+7=0)

13. Through the point *A(1,5)* is drawn a line parallel to the x-axis to meet at *B* the line *PQ* whose equation is $3y = 2x - 5$. Find the length of *AB* and the sine of the angle between *PQ* and *AB* ; hence show that the length of the perpendicular from *A* to *PQ* is $18/\sqrt{13}$. Calculate the area of the triangle formed by *PQ* and the axes . *(ans.:9,2/√13,25/12)*

14. Let $y = \frac{x^2+2}{x^2-1}$, express x in terms of y and find the values of y for which x is real . (ans.: $x = \mp\sqrt{\frac{y+2}{y-1}}; y \le -2 \ \ or \ \ y > 1$)

15. Find the domain and range of each function :

$$a)\ y = \frac{1}{1+x^2} \quad , \quad b)\ y = \frac{1}{1+\sqrt{x}} \quad , \quad c)\ y = \frac{1}{\sqrt{3-x}}$$

$(ans.: a)\forall x, 0 < y \le 1;\ b) x \ge 0, y > 0;\ c) x < 3, y > 0)$

16. Find the points of intersection of $x^2 = 4y$ and $y = 4x$. *(ans.:(0,0),(16,64))*

17. Find the coordinates of the points at which the curves cut the axes :

$a)\ y = x^3 - 9x^2 \quad , \quad b)\ y = (x^2-1)(x^2-9) \quad , \quad c)\ y = (x+1)(x-5)^2$

(ans.:a)(0,0);(0,0),(9,0);b)(0,9);(1,0),(-1,0),(3,0),(-3,0);c)(0,25);(-1,0),(5,0))

18. Let $f(x) = ax + b$ and $g(x) = cx + d$. What condition must be satisfied by the constants *a , b , c* and *d* to make *f(g(x))* and *g(f(x))* identical ?

(ans.:ad+b=bc+d)

19. A particle moves in the plane from *(-2,5)* to the y-axis in such away that $\Delta y = 3*\Delta x$. Find its new coordinates . *(ans.:(0,11),(0,-1))*

20. If $f(x) = 1/x$ and $g(x)=1/\sqrt{x}$, what are the domain of *f , g , f+g , f-g , f.g , f/g , g/f* , $f_o g$ and $g_o f$? What is the domain of $h(x) = g(x+4)$?

$(ans.: \forall x \ne 0, \forall x > 0, \forall x > 0, \forall x > 0, \forall x > 0, \forall x > 0, \forall x \ge 0, \forall x \ge 0, \forall x \ge 0; \forall x > -4)$

21. Discuss the continuity of :

$$f(x)=\begin{cases} x+\dfrac{1}{x} & for \quad x<0 \\ -x^3 & for \quad 0\le x<1 \\ -1 & for \quad 1\le x<2 \\ 1 & for \quad x=2 \\ 0 & for \quad x>2 \end{cases}$$

(ans.: discontinuous at x=0,2 ; continuous at x=1)

22. Evaluate the following limits :

a) $\lim\limits_{x\to\infty}\dfrac{x+Sinx}{2x+5}$ b) $\lim\limits_{x\to\infty}\dfrac{1+Sinx}{x}$

c) $\lim\limits_{x\to 0}\dfrac{x}{tan\,3x}$ d) $\lim\limits_{x\to\infty}\dfrac{x.Sinx}{(x+Sinx)^2}$

e) $\lim\limits_{x\to 1}\dfrac{1-\sqrt{x}}{1-x}$ f) $\lim\limits_{x\to 1}\dfrac{\sqrt{x+1}-\sqrt{2x}}{x^2-x}$

g) $\lim\limits_{n\to\infty}(\sqrt{n^2+1}-n)$

(ans.:a)1/2, b)0, c)1/3, d)0, e)1/2, f)-1/2√2, g)0)

23. Suppose that : $f(x)=x^3-3x^2-4x+12$ **and** $h(x)=\begin{cases}\dfrac{f(x)}{x-3} & for \;\; x\ne 3 \\ k & for \;\; x=3\end{cases}$.

Find : a) all zeros of f **.**

b) the value of k **that makes** h **continuous at** $x=3$ **.**

$(ans.: a)\, x=\mp 2,3;\; b)\, k=5)$