



3.2

Differentiation Rules

This section introduces a few rules that allow us to differentiate a great variety of functions. By proving these rules here, we can differentiate functions without having to apply the definition of the derivative each time.

Powers, Multiples, Sums, and Differences

The first rule of differentiation is that the derivative of every constant function is zero.

RULE 1 Derivative of a Constant Function

If f has the constant value $f(x) = c$, then

$$\frac{df}{dx} = \frac{d}{dx}(c) = 0.$$

EXAMPLE 1

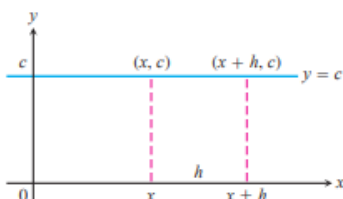


FIGURE 3.8 The rule $(d/dx)(c) = 0$ is another way to say that the values of constant functions never change and that the slope of a horizontal line is zero at every point.

If f has the constant value $f(x) = 8$, then

$$\frac{df}{dx} = \frac{d}{dx}(8) = 0.$$

Similarly,

$$\frac{d}{dx}\left(-\frac{\pi}{2}\right) = 0 \quad \text{and} \quad \frac{d}{dx}(\sqrt{3}) = 0. \quad \blacksquare$$

Proof of Rule 1 We apply the definition of derivative to $f(x) = c$, the function whose outputs have the constant value c (Figure 3.8). At every value of x , we find that

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{c - c}{h} = \lim_{h \rightarrow 0} 0 = 0. \quad \blacksquare$$



The second rule tells how to differentiate x^n if n is a positive integer.

RULE 2 Power Rule for Positive Integers

If n is a positive integer, then

$$\frac{d}{dx} x^n = nx^{n-1}.$$

To apply the Power Rule, we subtract 1 from the original exponent (n) and multiply the result by n .

EXAMPLE 2 Interpreting Rule 2

f	x	x^2	x^3	x^4	\dots
f'	1	$2x$	$3x^2$	$4x^3$	\dots

First Proof of Rule 2 The formula

$$z^n - x^n = (z - x)(z^{n-1} + z^{n-2}x + \dots + zx^{n-2} + x^{n-1})$$

can be verified by multiplying out the right-hand side. Then from the alternative form for the definition of the derivative,

$$\begin{aligned} f'(x) &= \lim_{z \rightarrow x} \frac{f(z) - f(x)}{z - x} = \lim_{z \rightarrow x} \frac{z^n - x^n}{z - x} \\ &= \lim_{z \rightarrow x} (z^{n-1} + z^{n-2}x + \dots + zx^{n-2} + x^{n-1}) \\ &= nx^{n-1} \end{aligned}$$

Second Proof of Rule 2 If $f(x) = x^n$, then $f(x + h) = (x + h)^n$. Since n is a positive integer, we can expand $(x + h)^n$ by the Binomial Theorem to get

$$\begin{aligned} f'(x) &= \lim_{h \rightarrow 0} \frac{f(x + h) - f(x)}{h} = \lim_{h \rightarrow 0} \frac{(x + h)^n - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{\left[x^n + nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots + nxh^{n-1} + h^n \right] - x^n}{h} \\ &= \lim_{h \rightarrow 0} \frac{nx^{n-1}h + \frac{n(n-1)}{2}x^{n-2}h^2 + \dots + nxh^{n-1} + h^n}{h} \\ &= \lim_{h \rightarrow 0} \left[nx^{n-1} + \frac{n(n-1)}{2}x^{n-2}h + \dots + nxh^{n-2} + h^{n-1} \right] \\ &= nx^{n-1} \end{aligned}$$

The third rule says that when a differentiable function is multiplied by a constant, its derivative is multiplied by the same constant.

**RULE 3 Constant Multiple Rule**

If u is a differentiable function of x , and c is a constant, then

$$\frac{d}{dx}(cu) = c \frac{du}{dx}.$$

In particular, if n is a positive integer, then

$$\frac{d}{dx}(cx^n) = cnx^{n-1}.$$

EXAMPLE 3

(a) The derivative formula

$$\frac{d}{dx}(3x^2) = 3 \cdot 2x = 6x$$

says that if we rescale the graph of $y = x^2$ by multiplying each y -coordinate by 3, then we multiply the slope at each point by 3 (Figure 3.9).

(b) A useful special case

The derivative of the negative of a differentiable function u is the negative of the function's derivative. Rule 3 with $c = -1$ gives

$$\frac{d}{dx}(-u) = \frac{d}{dx}(-1 \cdot u) = -1 \cdot \frac{d}{dx}(u) = -\frac{du}{dx}.$$

Proof of Rule 3

$$\frac{d}{dx}cu = \lim_{h \rightarrow 0} \frac{cu(x+h) - cu(x)}{h} \quad \text{Derivative definition with } f(x) = cu(x)$$

$$= c \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} \quad \text{Limit property}$$

$$= c \frac{du}{dx} \quad \text{u is differentiable.}$$

The next rule says that the derivative of the sum of two differentiable functions is the sum of their derivatives.

RULE 4 Derivative Sum Rule

If u and v are differentiable functions of x , then their sum $u + v$ is differentiable at every point where u and v are both differentiable. At such points,

$$\frac{d}{dx}(u + v) = \frac{du}{dx} + \frac{dv}{dx}.$$

**EXAMPLE 4** Derivative of a Sum

$$\begin{aligned}
 y &= x^4 + 12x \\
 \frac{dy}{dx} &= \frac{d}{dx}(x^4) + \frac{d}{dx}(12x) \\
 &= 4x^3 + 12
 \end{aligned}$$

Proof of Rule 4 We apply the definition of derivative to $f(x) = u(x) + v(x)$:

$$\begin{aligned}
 \frac{d}{dx}[u(x) + v(x)] &= \lim_{h \rightarrow 0} \frac{[u(x+h) + v(x+h)] - [u(x) + v(x)]}{h} \\
 &= \lim_{h \rightarrow 0} \left[\frac{u(x+h) - u(x)}{h} + \frac{v(x+h) - v(x)}{h} \right] \\
 &= \lim_{h \rightarrow 0} \frac{u(x+h) - u(x)}{h} + \lim_{h \rightarrow 0} \frac{v(x+h) - v(x)}{h} = \frac{du}{dx} + \frac{dv}{dx}.
 \end{aligned}$$

Combining the Sum Rule with the Constant Multiple Rule gives the **Difference Rule**, which says that the derivative of a *difference* of differentiable functions is the difference of their derivatives.

$$\frac{d}{dx}(u - v) = \frac{d}{dx}[u + (-1)v] = \frac{du}{dx} + (-1)\frac{dv}{dx} = \frac{du}{dx} - \frac{dv}{dx}$$

The Sum Rule also extends to sums of more than two functions, as long as there are only finitely many functions in the sum. If u_1, u_2, \dots, u_n are differentiable at x , then so is $u_1 + u_2 + \dots + u_n$, and

$$\frac{d}{dx}(u_1 + u_2 + \dots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \dots + \frac{du_n}{dx}.$$

EXAMPLE 5 Derivative of a Polynomial

$$\begin{aligned}
 y &= x^3 + \frac{4}{3}x^2 - 5x + 1 \\
 \frac{dy}{dx} &= \frac{d}{dx}x^3 + \frac{d}{dx}\left(\frac{4}{3}x^2\right) - \frac{d}{dx}(5x) + \frac{d}{dx}(1) \\
 &= 3x^2 + \frac{4}{3} \cdot 2x - 5 + 0 \\
 &= 3x^2 + \frac{8}{3}x - 5
 \end{aligned}$$

Notice that we can differentiate any polynomial term by term, the way we differentiated the polynomial in Example 5. All polynomials are differentiable everywhere.

Proof of the Sum Rule for Sums of More Than Two Functions We prove the statement

$$\frac{d}{dx}(u_1 + u_2 + \dots + u_n) = \frac{du_1}{dx} + \frac{du_2}{dx} + \dots + \frac{du_n}{dx}$$

by mathematical induction (see Appendix 1). The statement is true for $n = 2$, as was just proved. This is Step 1 of the induction proof.



Step 2 is to show that if the statement is true for any positive integer $n = k$, where $k \geq n_0 = 2$, then it is also true for $n = k + 1$. So suppose that

$$\frac{d}{dx}(u_1 + u_2 + \cdots + u_k) = \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx}. \quad (1)$$

Then

$$\begin{aligned} & \frac{d}{dx}(u_1 + u_2 + \cdots + u_k + u_{k+1}) \\ & \quad \text{Call the function defined by this sum } u. \quad \text{Call this function } v. \\ &= \frac{d}{dx}(u_1 + u_2 + \cdots + u_k) + \frac{du_{k+1}}{dx} \quad \text{Rule 4 for } \frac{d}{dx}(u + v) \\ &= \frac{du_1}{dx} + \frac{du_2}{dx} + \cdots + \frac{du_k}{dx} + \frac{du_{k+1}}{dx}. \quad \text{Eq. (1)} \end{aligned}$$

With these steps verified, the mathematical induction principle now guarantees the Sum Rule for every integer $n \geq 2$. ■

EXAMPLE 6 Finding Horizontal Tangents

Does the curve $y = x^4 - 2x^2 + 2$ have any horizontal tangents? If so, where?

Solution The horizontal tangents, if any, occur where the slope dy/dx is zero. We have,

$$\frac{dy}{dx} = \frac{d}{dx}(x^4 - 2x^2 + 2) = 4x^3 - 4x.$$

Now solve the equation $\frac{dy}{dx} = 0$ for x :

$$\begin{aligned} 4x^3 - 4x &= 0 \\ 4x(x^2 - 1) &= 0 \\ x &= 0, 1, -1. \end{aligned}$$

The curve $y = x^4 - 2x^2 + 2$ has horizontal tangents at $x = 0, 1$, and -1 . The corresponding points on the curve are $(0, 2)$, $(1, 1)$ and $(-1, 1)$. See Figure 3.10. ■

Products and Quotients

While the derivative of the sum of two functions is the sum of their derivatives, the derivative of the product of two functions is *not* the product of their derivatives. For instance,

$$\frac{d}{dx}(x \cdot x) = \frac{d}{dx}(x^2) = 2x, \quad \text{while} \quad \frac{d}{dx}(x) \cdot \frac{d}{dx}(x) = 1 \cdot 1 = 1.$$

The derivative of a product of two functions is the sum of *two* products, as we now explain.

RULE 5 Derivative Product Rule

If u and v are differentiable at x , then so is their product uv , and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}.$$