Chapter two
Functions

2-1- Exponential and Logarithm functions :

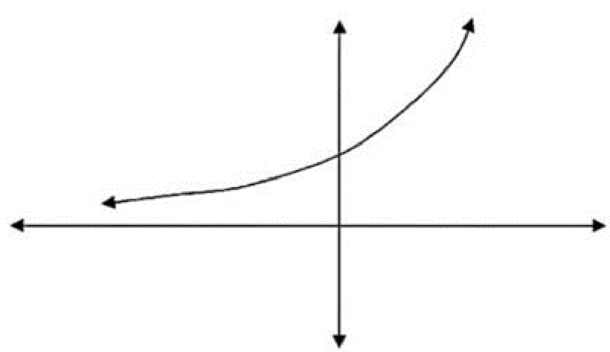
Exponential functions : If a is a positive number and x is any number , we define the exponential function as :

$y = a^x$ with domain : $-\infty < x < \infty$
Range : $y > 0$

The properties of the exponential functions are :

1. If $a > 0 \leftrightarrow a^x > 0$.
2. $a^x . a^y = a^{x+y}$.
3. $a^x / a^y = a^{x-y}$.
4. $(a^x)^y = a^{x.y}$.
5. $(a . b)^x = a^x . b^x$.
6. $a^{\frac{x}{y}} = \sqrt[y]{a^x} = (\sqrt[y]{a})^x$.
7. $a^{-x} = 1 / a^x$ and $a^x = 1 / a^{-x}$.
8. $a^x = a^y \leftrightarrow x = y$.
9. $a^0 = 1$,
   $a^{\infty} = \infty$, $a^{-\infty} = 0$, where $a > 1$.
   $a^{\infty} = 0$, $a^{-\infty} = \infty$, where $a < 1$.

The graph of the exponential function $y = a^x$ is :



Logarithm function : If a is any positive number other than 1 , then the logarithm of x to the base a denoted by :

$y = \log_a x$ where $x > 0$

At a = e = 2.7182828… , we get the natural logarithm and denoted by :

$y = \ln x$

Let $x , y > 0$ then the properties of logarithm functions are :

1. $y = a^x \leftrightarrow x = \log_a y$ and $y = e^x \leftrightarrow x = \ln y$.
2. $\log_e x = \ln x$.
3. $\log_a x = \ln x / \ln a$.

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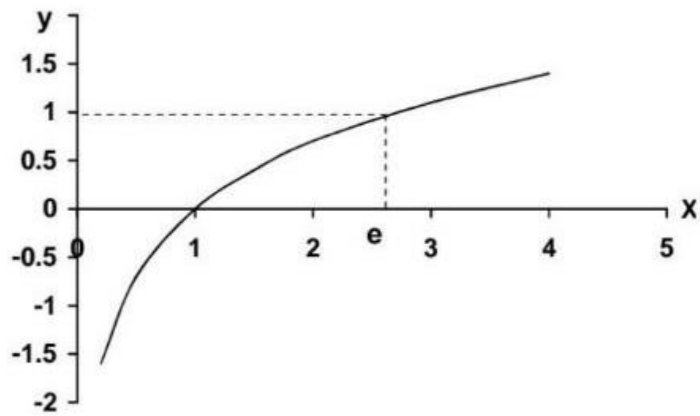
4. $\ln (x.y) = \ln x + \ln y$.
5. $\ln (x / y) = \ln x - \ln y$.
6. $\ln x^n = n. \ln x$.
7. $\ln e = \log_a a = 1$ and $\ln 1 = \log_a 1 = 0$.
8. $a^x = e^{x. \ln a}$.
9. $e^{\ln x} = x$.

The graph of the function $y = \ln x$ is :



Application of exponential and logarithm functions :

We take Newton's law of cooling :

$$T - T_S = (T_0 - T_S) e^{t k}$$

where T is the temperature of the object at time t .
T_S is the surrounding temperature .
T_0 is the initial temperature of the object .
k is a constant .

EX-1- The temperature of an ingot of metal is $80\ ^oC$ and the room temperature is $20\ ^oC$. After twenty minutes, it was $70\ ^oC$.

a) What is the temperature will the metal be after 30 minutes?
b) What is the temperature will the metal be after two hours?
c) When will the metal be $30\ ^oC$?

Sol. :

$$T - T_S = (T_0 - T_S)e^{tk} \Rightarrow 50 = 60e^{20k} \Rightarrow k = \frac{\ln 5 - \ln 6}{20} = -0.0091$$

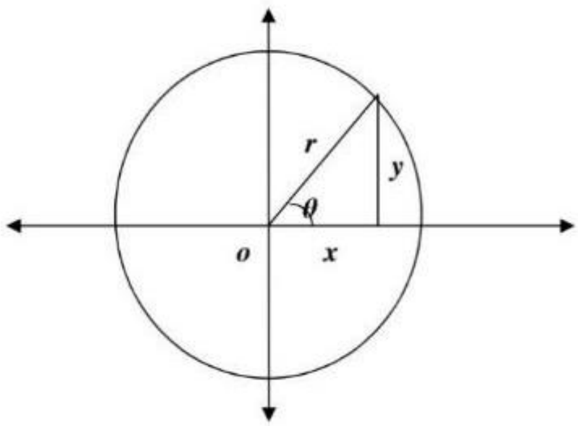
$$a)\quad T - 20 = 60e^{30(-0.0091)} = 60 * 0.761 = 45.6\ ^oC \Rightarrow T = 65.6\ ^oC$$

$$b)\quad T - T_S = 60e^{120(-0.0091)} = 60 * 0.335 = 20.1\ ^oC \Rightarrow T = 40.1\ ^oC$$

$$c)\quad 10 = 60e^{-0.0091\, t} \Rightarrow -0.0091\, t = -\ln 6 \Rightarrow t = 3.3\ hrs.$$

2-2- Trigonometric functions : When an angle of measure θ is placed in standard position at the center of a circle of radius r , the trigonometric functions of θ are defined by the equations :

$$Sin\theta=\frac{y}{r}=\frac{1}{csc\,\theta} \quad , \quad Cos\theta=\frac{x}{r}=\frac{1}{sec\,\theta} \quad , \quad tan\,\theta=\frac{y}{x}=\frac{1}{Cot\theta}=\frac{Sin\theta}{Cos\theta}$$



The following are some properties of these functions :

1) $Sin^2\theta+Cos^2\theta=1$

2) $1+tan^2\theta=sec^2\theta$ and $1+Cot^2\theta=csc^2\theta$

3) $Sin(\theta\mp\beta)=Sin\theta.Cos\beta\mp Cos\theta.Sin\beta$

4) $Cos(\theta\mp\beta)=Cos\theta.Cos\beta\pm Sin\theta.Sin\beta$

5) $tan(\theta\mp\beta)=\dfrac{tan\,\theta\mp tan\,\beta}{1\pm tan\,\theta.tan\,\beta}$

6) $Sin2\theta=2Sin\theta.Cos\theta$ and $Cos2\theta=Cos^2\theta-Sin^2\theta$

7) $Cos^2\theta=\dfrac{1+Cos2\theta}{2}$ and $Sin^2\theta=\dfrac{1-Cos2\theta}{2}$

8) $Sin(\theta\mp\frac{\pi}{2})=\mp Cos\theta$ and $Cos(\theta\mp\frac{\pi}{2})=\pm Sin\theta$

9) $Sin(-\theta)=-Sin\theta$ and $Cos(-\theta)=Cos\theta$ and $tan(-\theta)=-tan\,\theta$

10) $Sin\theta.Sin\beta=\frac{1}{2}[Cos(\theta-\beta)-Cos(\theta+\beta)]$

$Cos\theta.Cos\beta=\frac{1}{2}[Cos(\theta-\beta)+Cos(\theta+\beta)]$

$Sin\theta.Cos\beta=\frac{1}{2}[Sin(\theta-\beta)+Sin(\theta+\beta)]$

11) $Sin\theta + Sin\beta = 2\,Sin\dfrac{\theta+\beta}{2}.Cos\dfrac{\theta-\beta}{2}$

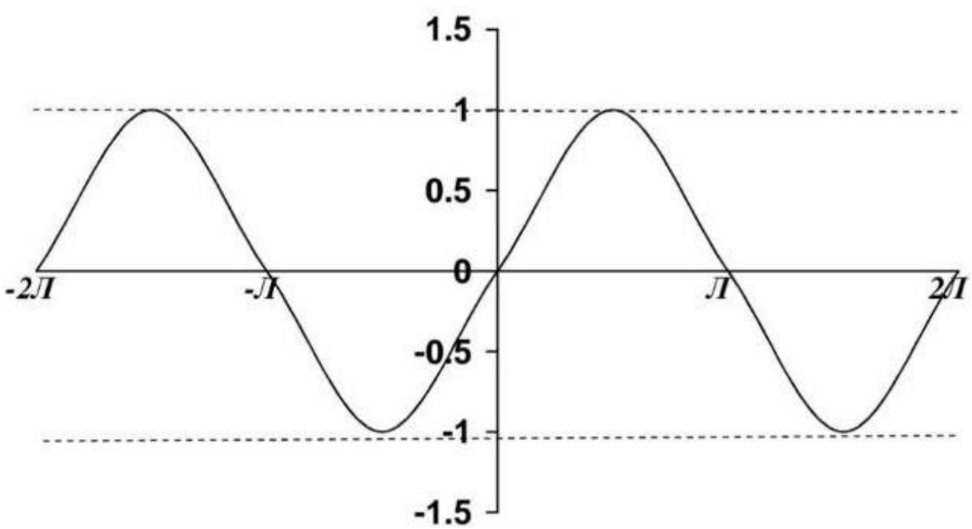
$Sin\theta - Sin\beta = 2Cos\dfrac{\theta+\beta}{2}.Sin\dfrac{\theta-\beta}{2}$

12) $Cos\theta + Cos\beta = 2Cos\dfrac{\theta+\beta}{2}.Cos\dfrac{\theta-\beta}{2}$

$Cos\theta - Cos\beta = -2\,Sin\dfrac{\theta+\beta}{2}.Sin\dfrac{\theta-\beta}{2}$

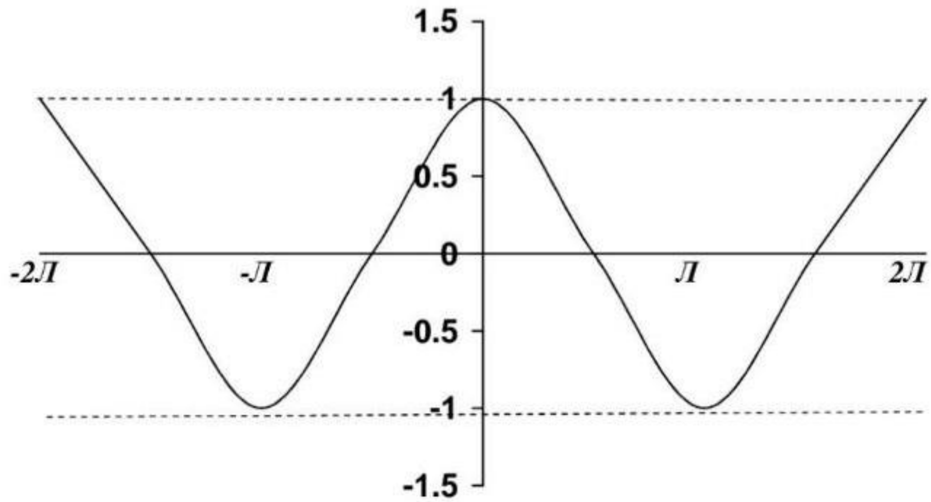
θ	0	$\Pi/6$	$\Pi/4$	$\Pi/3$	$\Pi/2$	Π
$Sin\theta$	0	$1/2$	$1/\sqrt{2}$	$\sqrt{3}/2$	1	0
$Cos\theta$	1	$\sqrt{3}/2$	$1/\sqrt{2}$	$1/2$	0	-1
$tan\theta$	0	$1/\sqrt{2}$	1	$\sqrt{3}$	∞	0

Graphs of the trigonometric functions are :

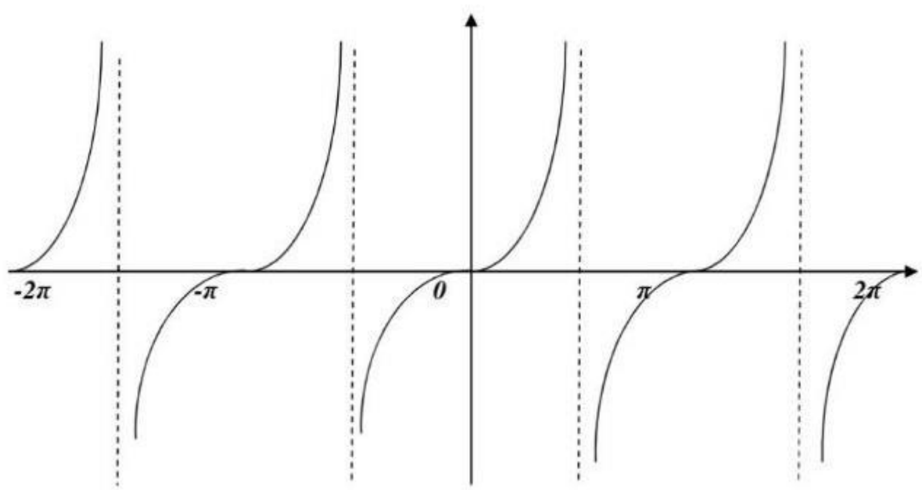


$y = Sinx \quad D_x : \forall x$

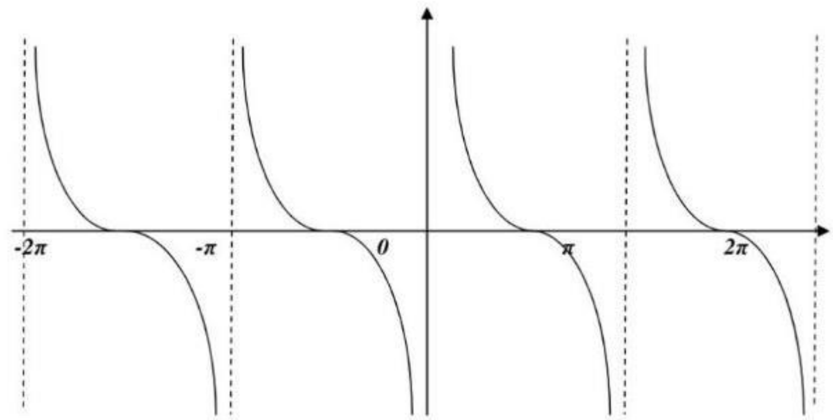
$R_y : -1 \le y \le 1$



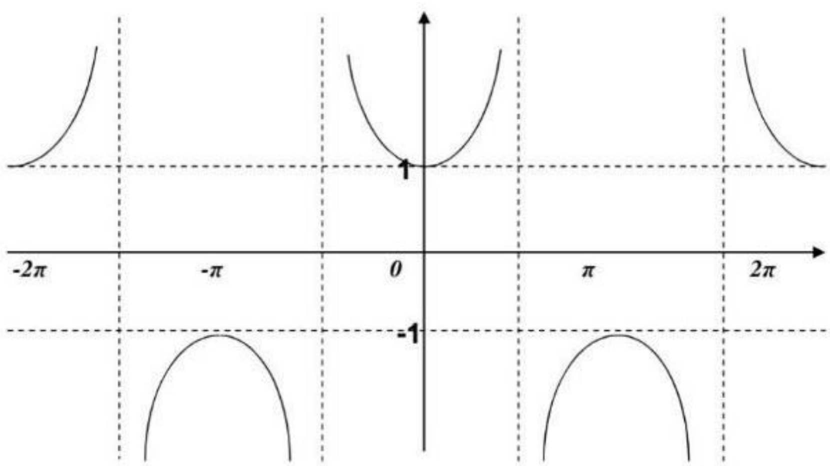
$$y = Cos x \quad D_x : \forall x$$
$$R_y : -1 \le y \le 1$$



$$y = \tan x \quad D_x : \forall x \ne \frac{2n+1}{2}\pi$$
$$R_y : \forall y$$



$$y = Cotx \quad D_x : \forall x \neq n\pi$$
$$R_y : \forall y$$



$$y = Secx \quad D_x : \forall x \neq \frac{2n+1}{2}\pi$$
$$R_y : \forall y \geq 1 \ \ or \ \ y \leq -1$$

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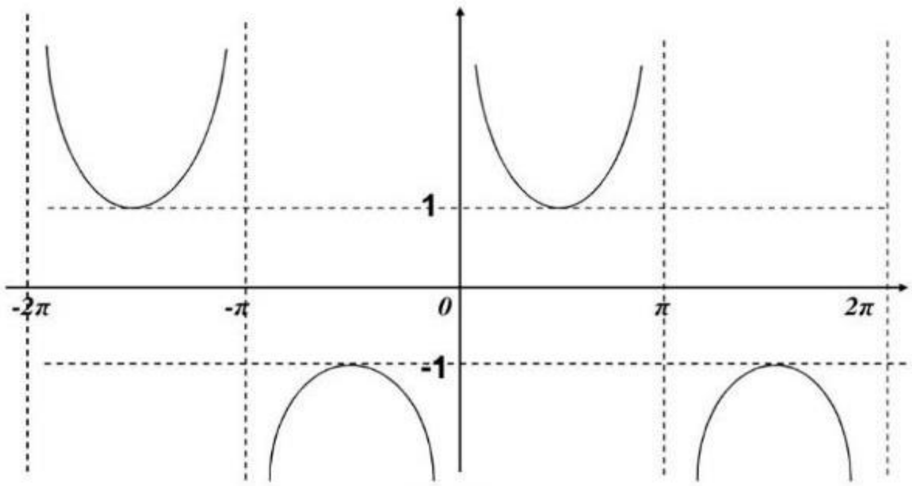


$y = \csc x \quad D_x : \forall x \neq n\pi$

$R_y : \forall y \geq 1 \text{ or } y \leq -1$

Where $n = 0, \mp 1, \mp 2, \mp 3, \ldots\ldots$

EX-2 - Solve the following equations , for values of θ from 0^o to 360^o inclusive .

a) $\tan\theta = 2\,Sin\,\theta$ b) $1 + Cos\,\theta = 2\,Sin^2\,\theta$

Sol.-

a) $\tan\theta = 2Sin\theta \Rightarrow \frac{Sin\theta}{Cos\theta} = 2Sin\theta$

$\Rightarrow Sin\theta(1 - 2Cos\theta) = 0$

either $Sin\theta = 0 \Rightarrow \theta = 0^o, 180^o, 360^o$

or $Cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$

Therefore the required values of θ are $0^o, 60^o, 180^o, 300^o, 360^o$.

b) $1 + Cos\theta = 2.Sin^2\theta \Rightarrow 1 + Cos\theta = 2(1 - Cos^2\theta)$

$\Rightarrow (2Cos\theta - 1)(Cos\theta + 1) = 0$

either $Cos\theta = \frac{1}{2} \Rightarrow \theta = 60^o, 300^o$

or $Cos\theta = -1 \Rightarrow \theta = 180^o$

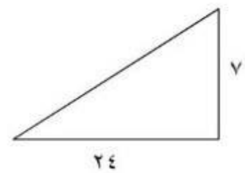
There the roots of the equation between 0^o and 360^o are $60^o, 180^o$ and 300^o .

EX-3- If $tan\ \theta = 7/24$, find without using tables the values of $Sec\theta$ and $Sin\theta$.

Sol.-

$$tan\,\theta = \frac{y}{x} = \frac{7}{24} \Rightarrow r = \sqrt{7^2 + 24^2} = 25$$

$$Sec\,\theta = \frac{r}{x} = \frac{25}{24} \quad and \quad Sin\,\theta = \frac{y}{r} = \frac{7}{25}$$



EX-4- Prove the following identities :

a) $Csc\theta + tan\,\theta.Sec\theta = Csc\theta.Sec^2\theta$

b) $Cos^4\theta - Sin^4\theta = Cos^2\theta - Sin^2\theta$

c) $\dfrac{Sec\theta - Csc\theta}{tan\,\theta - Cot\theta} = \dfrac{tan\,\theta + Cot\theta}{Sec\theta + Csc\theta}$

Sol.-

a) $$L.H.S. = Csc\theta + tan\,\theta.Sec\theta = \frac{1}{Sin\theta} + \frac{Sin\theta}{Cos\theta}.\frac{1}{Cos\theta}$$

$$= \frac{Cos^2\theta + Sin^2\theta}{Sin\theta.Cos^2\theta} = \frac{1}{Sin\theta}.\frac{1}{Cos^2\theta} = Csc\theta.Sec^2\theta = R.H.S.$$

b) $$L.H.S. = Cos^4\theta - Sin^4\theta = (Cos^2\theta - Sin^2\theta).(Cos^2\theta + Sin^2\theta)$$

$$= Cos^2\theta - Sin^2\theta = R.H.S.$$

c) $$L.H.S. = \frac{Sec\theta - Csc\theta}{tan\,\theta - Cot\theta} = \frac{\frac{1}{Cos\theta} - \frac{1}{Sin\theta}}{\frac{Sin\theta}{Cos\theta} - \frac{Cos\theta}{Sin\theta}} = \frac{1}{Sin\theta + Cos\theta}$$

$$= \frac{Sin^2\theta + Cos^2\theta}{Sin\theta + Cos\theta}.\frac{\frac{1}{Sin\theta.Cos\theta}}{\frac{1}{Sin\theta.Cos\theta}} = \frac{tan\,\theta + Cot\theta}{Sec\theta + Csc\theta} = R.H.S.$$

EX-5- Simplify $\dfrac{1}{\sqrt{x^2 - a^2}}$ when $x = a.Csc\theta$.

Sol.- $$\frac{1}{\sqrt{x^2 - a^2}} = \frac{1}{\sqrt{a^2Csc^2\theta - a^2}} = \frac{1}{a\sqrt{Cot^2\theta}} = \frac{1}{a}tan\,\theta\ .$$

EX-6- Eliminate θ from the equations :

i) $x = a\ Sin\theta$ and $y = b\ tan\theta$

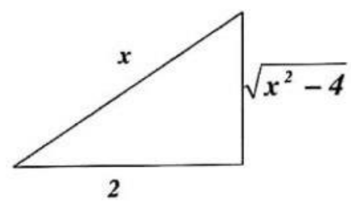
ii) $x = 2\ Sec\theta$ and $y = Cos2\theta$

Sol.-

i) $x = a.Sin\theta \Rightarrow Sin\theta = \frac{x}{a} \Rightarrow Csc\theta = \frac{a}{x}$

$y = b\tan\theta \Rightarrow \tan\theta = \frac{y}{b} \Rightarrow Cot\theta = \frac{b}{y}$

Since $Csc^2\theta = Cot^2\theta + 1 \Rightarrow \frac{a^2}{x^2} = \frac{b^2}{y^2} + 1$



ii) $x = 2\,Sec\,\theta \Rightarrow Cos\,\theta = \frac{2}{x}$

$y = Cos\,2\theta \Rightarrow y = Cos^2\theta - Sin^2\theta$

$y = \frac{4}{x^2} - \frac{x^2-4}{x^2} \Rightarrow x^2 y = 8 - x^2$

EX-7- If $\tan^2\theta - 2\tan^2\beta = 1$, show that $2\,Cos^2\theta - Cos^2\beta = 0$.

Sol. –

$\tan^2\theta - 2\tan^2\beta = 1 \Rightarrow Sec^2\theta - 1 - 2(Sec^2\beta - 1) = 1$

$\Rightarrow Sec^2\theta - 2Sec^2\beta = 0 \Rightarrow \frac{1}{Cos^2\theta} - \frac{2}{Cos^2\beta} = 0$

$\Rightarrow 2Cos^2\theta - Cos^2\beta = 0$ Q.E.D.

EX-8- If $a\,Sin\theta = p - b\,Cos\theta$ and $b\,Sin\theta = q + a\,Cos\theta$ .Show that :
$a^2 + b^2 = p^2 + q^2$

Sol.-

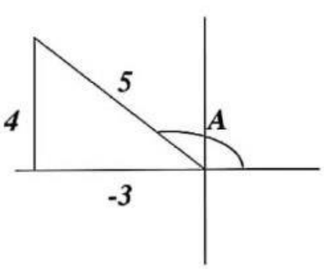
$p = a.Sin\theta + b.Cos\theta$ and $q = b.Sin\theta - a.Cos\theta$

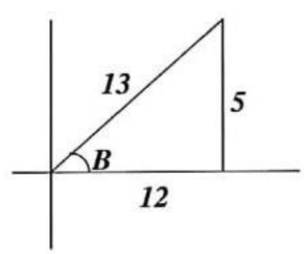
$p^2 + q^2 = (aSin\theta + bCos\theta)^2 + (bSin\theta - aCos\theta)^2$

$= a^2(Sin^2\theta + Cos^2\theta) + b^2(Cos^2\theta + Sin^2\theta) = a^2 + b^2$

EX-9- If $Sin\ A = 4/5$ and $Cos\ B = 12/13$,where A is obtuse and B is acute . Find , without tables , the values of :
a) Sin (A – B) , b) tan (A – B) , c) $\tan(A+B)$.

Sol. -





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$$a)\quad Sin(A-B)=SinA.CosB-CosA.SinB$$
$$=\frac{4}{5}.\frac{12}{13}+\frac{3}{5}.\frac{5}{13}=\frac{63}{65}$$

$$b)\quad tan(A-B)=\frac{tan\,A-tan\,B}{1+tan\,A.tan\,B}$$
$$=\frac{-\frac{4}{3}-\frac{5}{12}}{1-\frac{4}{3}.\frac{5}{12}}=-\frac{63}{16}$$

$$c)\quad tan(A+B)=\frac{tan\,A+tan\,B}{1-tan\,A.tan\,B}$$
$$=\frac{-\frac{4}{3}+\frac{5}{12}}{1+\frac{4}{3}.\frac{5}{12}}=\frac{33}{56}$$

EX-10 – **Prove the following identities:**

$$a)\quad Sin(A+B)+Sin(A-B)=2.SinA.CosB$$
$$b)\quad tan\,A+tan\,B=\frac{Sin(A+B)}{CosA.CosB}$$
$$c)\quad Sec(A+B)=\frac{SecA.SecB.CscA.CscB}{CscA.CscB-SecA.SecB}$$
$$d)\quad \frac{Sin2\theta+Cos2\theta+1}{Sin2\theta-Cos2\theta+1}=Cot\theta$$

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Sol.-

a) $L.H.S. = Sin(A+B) + Sin(A-B)$

$= SinA.CosB + CosA.SinB + SinA.CosB - CosA.SinB$

$= 2.SinA.CosB = R.H.S.$

b) $R.H.S. = \dfrac{Sin(A+B)}{CosA.CosB} = \dfrac{SinA.CosB + CosA.SinB}{CosA.CosB}$

$= tan\,A + tan\,B = L.H.S.$

c) $R.H.S = \dfrac{SecA.SecB.CscA.CscB}{CscA.CscB - SecA.SecB} = \dfrac{\frac{1}{CosA}.\frac{1}{CosB}.\frac{1}{SinA}.\frac{1}{SinB}}{\frac{1}{SinA}.\frac{1}{SinB} - \frac{1}{CosA}.\frac{1}{CosB}}$

$= \dfrac{1}{CosA.CosB - SinA.SinB} = \dfrac{1}{Cos(A+B)}$

$= Sec(A+B) = L.H.S.$

d) $L.H.S. = \dfrac{Sin2\theta + Cos2\theta + 1}{Sin2\theta - Cos2\theta + 1} = \dfrac{2Sin\theta.Cos\theta + (Cos^2\theta - Sin^2\theta) + 1}{2Sin\theta.Cos\theta - (Cos^2\theta - Sin^2\theta) + 1}$

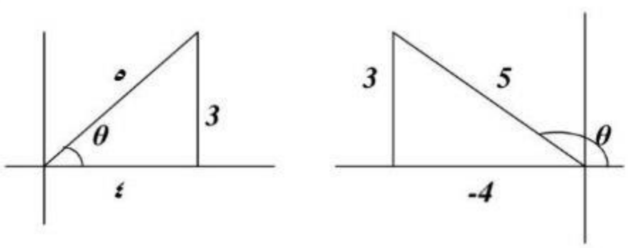
$= \dfrac{2Sin\theta.Cos\theta + 2Cos^2\theta}{2Sin\theta.Cos\theta + 2Sin^2\theta} = \dfrac{Cos\theta}{Sin\theta} = Cot\theta = R.H.S.$

EX-11 – Find , without using tables , the values of $Sin\,2\theta$ and $Cos\,2\theta$, when:

a) $Sin\theta = 3/5$, b) $Cos\,\theta = 12/13$, c) $Sin\,\theta = -\sqrt{3}/2$.

Sol. –

a)



$Sin2\theta = 2.Sin\theta.Cos\theta = 2.\frac{3}{5}.(\mp\frac{4}{5}) = \mp\frac{24}{25}$

$Cos2\theta = Cos^2\theta - Sin^2\theta = (\mp\frac{4}{5})^2 - (\frac{3}{5})^2 = \frac{7}{25}$

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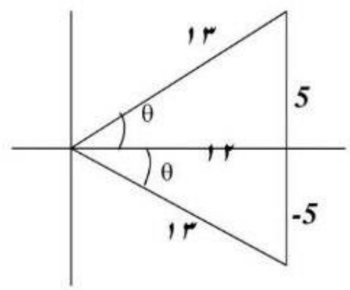
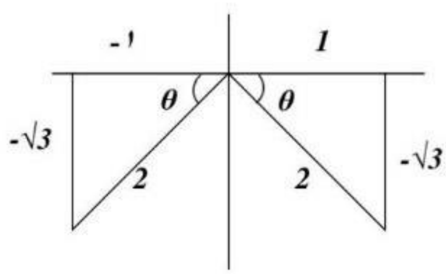
b)



$$Sin2\theta = 2.Sin\theta.Cos\theta = 2(\mp\frac{5}{13}).(\frac{12}{13}) = \mp\frac{120}{169}$$

$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\frac{12}{13})^2 - (\mp\frac{5}{13})^2 = \frac{119}{169}$$

c)



$$Sin2\theta = 2Sin\theta.Cos\theta = 2(-\frac{\sqrt{3}}{2}).(\mp\frac{1}{2}) = \pm\frac{\sqrt{3}}{2}$$

$$Cos2\theta = Cos^2\theta - Sin^2\theta = (\mp\frac{1}{2})^2 - (-\frac{\sqrt{3}}{2})^2 = -\frac{1}{2}$$

EX-12- **Solve the following equations for values of θ from 0^o to 360^o inclusive:**

a) $Cos\ 2\theta + Cos\ \theta + 1 = 0$, **b)** $4\tan\theta . \tan 2\theta = 1$

Sol.-

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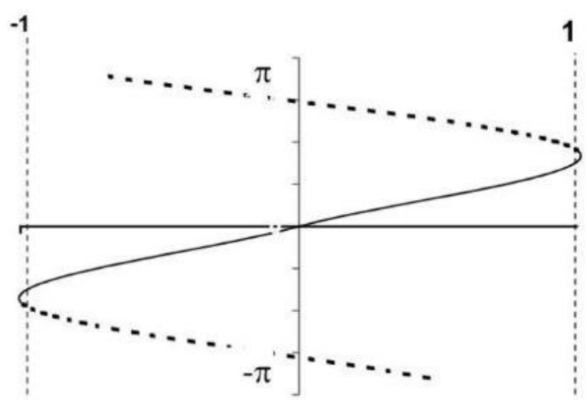
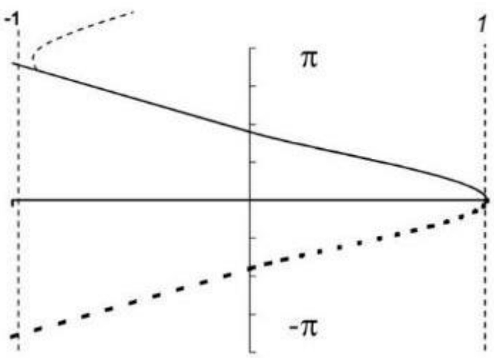
$$a)\quad Cos2\theta + Cos\theta + 1 = 0 \Rightarrow 2Cos^2\theta - 1 + Cos\theta + 1 = 0$$
$$\Rightarrow Cos(2.Cos\theta + 1) = 0$$
$$either \quad Cos\theta = 0 \Rightarrow \theta = 90^o, 270^o$$
$$or \quad Cos\theta = -\frac{1}{2} \Rightarrow \theta = 120^o, 240^o$$
$$\theta = \{90^o, 120^o, 240^o, 270^o\}$$
$$b)\quad 4.\tan\theta.\tan 2\theta = 1 \Rightarrow 4.\tan\theta.\frac{2\tan\theta}{1-\tan^2\theta} = 1$$
$$\Rightarrow 9\tan^2\theta = 1$$
$$either \quad \tan\theta = \frac{1}{3} \Rightarrow \theta = 18.4^o, 198.4^o$$
$$or \quad \tan\theta = -\frac{1}{3} \Rightarrow \theta = 161.6^o, 341.6^o$$
$$\theta = \{18.4^o, 161.6^o, 198.4^o, 341.6^o\}$$

2-3- <u>The inverse trigonometric functions</u> : The inverse trigonometric functions arise in problems that require finding angles from side measurements in triangles :

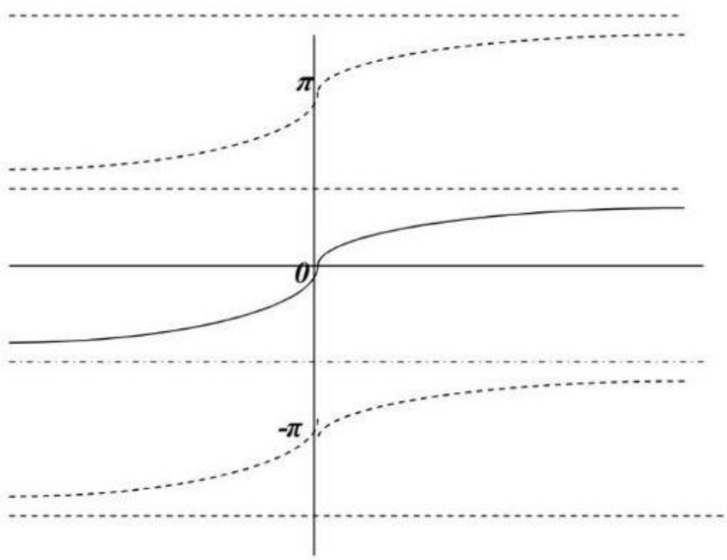
$$y = Sinx \quad \Leftrightarrow \quad x = Sin^{-1}y$$



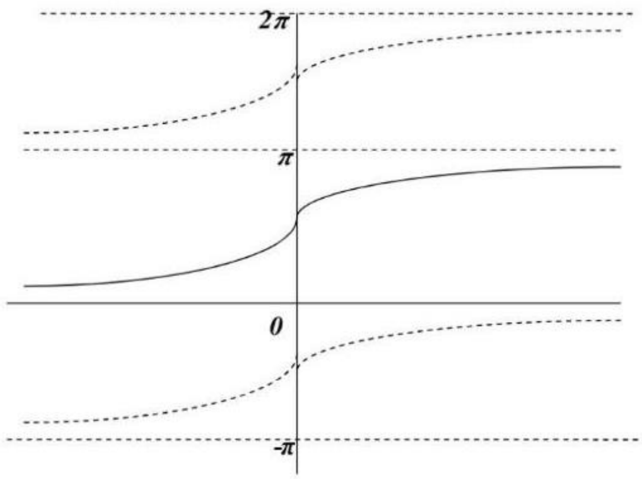
$$y = Sin^{-1}x \quad D_x : -1 \le x \le 1$$
$$R_y : -90 \le y \le 90$$



$$y = Cos^{-1}x \quad D_x : -1 \le x \le 1$$
$$R_y : 0 \le y \le 180$$

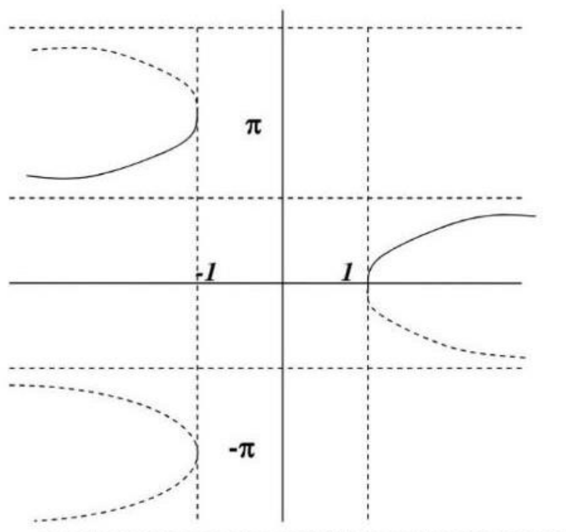


$$y = tan^{-1}x \quad D_x : \forall x$$
$$R_y : -90 \le y \le 90$$



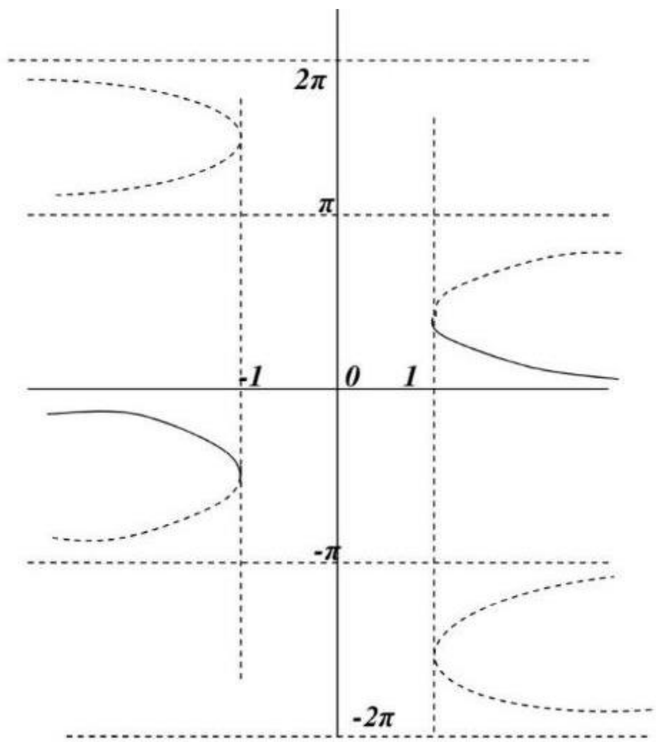
$y = Cot^{-1}x \quad D_x : \forall x$

$R_y : 0 \le y \le \pi$



$y = Sec^{-1}x \quad D_x : \forall |x| \ge 1$

$R_y : 0 \le y \le \pi, y \ne \frac{\pi}{2}$



$$y = Csc^{-1}x \quad D_x : \forall |x| \geq 1$$

$$R_y : -\frac{\pi}{2} \leq y \leq \frac{\pi}{2}, y \neq 0$$

The following are some properties of the inverse trigonometric functions :

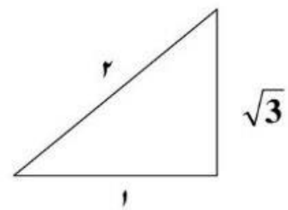
1. $Sin^{-1}(-x) = -Sin^{-1}x$
2. $Cos^{-1}(-x) = \pi - Cos^{-1}x$
3. $Sin^{-1}x + Cos^{-1}x = \frac{\pi}{2}$
4. $tan^{-1}(-x) = -tan^{-1}x$
5. $Cot^{-1}x = \frac{\pi}{2} - tan^{-1}x$
6. $Sec^{-1}x = Cos^{-1}\frac{1}{x}$
7. $Csc^{-1}x = Sin^{-1}\frac{1}{x}$
8. $Sec^{-1}(-x) = \pi - Sec^{-1}x$

$$and \ noted \ that \ (Sinx)^{-1} = \frac{1}{Sinx} = Cscx \neq Sin^{-1}x$$

EX-13- Given that $\alpha = Sin^{-1}\dfrac{\sqrt{3}}{2}$, find :

$Csc\alpha \ , Cos\alpha \ , Sec\alpha \ , tan\,\alpha \ , and \ , Cot\alpha$

Sol.-



$$\alpha = Sin^{-1}\frac{\sqrt{3}}{2} \Rightarrow Sin\alpha = \frac{\sqrt{3}}{2} = \frac{x}{y} \Rightarrow r = \sqrt{4-3} = 1$$

$$Csc\alpha = \frac{2}{\sqrt{3}} \ , Cos\alpha = \frac{1}{2} \ , Sec\alpha = 2 \ , tan\,\alpha = \sqrt{3} \ , Cot\alpha = \frac{1}{\sqrt{3}}$$

EX-14 – Evaluate the following expressions :

$$a)\ Sec(Cos^{-1}\frac{1}{2}) \quad b)\ Sin^{-1}1 - Sin^{-1}(-1) \quad c)\ Cos^{-1}(-Sin\frac{\pi}{6})$$

Sol.-

$$a)\ Sec(Cos^{-1}\frac{1}{2}) = Sec\frac{\pi}{3} = 2$$

$$b)\ Sin^{-1}1 - Sin^{-1}(-1) = \frac{\pi}{2} - (-\frac{\pi}{2}) = \pi$$

$$c)\ Cos^{-1}(-Sin\frac{\pi}{6}) = Cos^{-1}(-\frac{1}{2}) = \frac{2}{3}\pi$$

EX-15- Prove that :

$$a)\ Sec^{-1}x = Cos^{-1}\frac{1}{x} \qquad b)\ Sin^{-1}(-x) = -Sin^{-1}x$$

Sol.

$$a)\quad Let\ \ y = Sec^{-1}x \Rightarrow x = Secy \Rightarrow x = \frac{1}{Cosy}$$

$$\Rightarrow y = Cos^{-1}\frac{1}{x} \Rightarrow Sec^{-1}x = Cos^{-1}\frac{1}{x}$$

$$b)\quad Let\ \ y = -Sin^{-1}x \Rightarrow x = Sin(-y) \Rightarrow x = -Siny$$

$$\Rightarrow y = Sin^{-1}(-x) \Rightarrow Sin^{-1}(-x) = -Sin^{-1}x$$

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