Chapter four

Applications of derivatives

4-1- L'Hopital rule :

Suppose that $f(x_o) = g(x_o) = 0$ and that the functions f and g are both differentiable on an open interval (a, b) that contains the point x_o. Suppose also that $g'(x) \neq 0$ at every point in (a, b) except possibly x_o. Then :

$$\lim_{x \to x_o} \frac{f(x)}{g(x)} = \lim_{x \to x_o} \frac{f'(x)}{g'(x)} \quad \text{provided the limit exists .}$$

Differentiate f and g as long as you still get the form $\frac{0}{0}$ or $\frac{\infty}{\infty}$ at $x = x_o$. Stop differentiating as soon as you get something else . L'Hopital's rule does not apply when either the numerator or denominator has a finite non-zero limit .

EX-1 – Evaluate the following limits :

$$1)\ \lim_{x \to 0} \frac{\sin x}{x} \qquad 2)\ \lim_{x \to 2} \frac{\sqrt{x^2+5}-3}{x^2-4}$$

$$3)\ \lim_{x \to 0} \frac{x - \sin x}{x^3} \qquad 4)\ \lim_{x \to \frac{\pi}{2}} -\left(x - \frac{\pi}{2}\right).\tan x$$

Sol. –

$$1)\ \lim_{x \to 0} \frac{\sin x}{x} \Rightarrow \frac{0}{0} \quad using\ L'Hoptal's\ rule \Rightarrow$$

$$= \lim_{x \to 0} \frac{\cos x}{1} = \cos 0 = 1$$

$$2)\ \lim_{x \to 2} \frac{\sqrt{x^2+5}-3}{x^2-4} \Rightarrow \frac{0}{0} \quad using\ L'Hoptal's\ rule \Rightarrow$$

$$= \lim_{x \to 2} \frac{\frac{x}{\sqrt{x^2+5}}}{2x} = \lim_{x \to 2} \frac{1}{2\sqrt{x^2+5}} = \frac{1}{2\sqrt{4+5}} = \frac{1}{6}$$

$$3)\ \lim_{x \to 0} \frac{x - \sin x}{x^3} \Rightarrow \frac{0}{0} \quad using\ L'Hoptal's\ rule \Rightarrow$$

$$= \lim_{x \to 0} \frac{1 - \cos x}{3x^2} \Rightarrow \frac{0}{0} \quad using\ L'Hopital's\ rule \Rightarrow$$

$$= \frac{1}{6} \lim_{x \to 0} \frac{\sin x}{x} = \frac{1}{6}$$

$$4)\quad \lim_{x\to\frac{\pi}{2}} -\left(x-\frac{\pi}{2}\right)\tan x \Rightarrow 0.\infty \ \text{ we can't using L'Hoptal's rule} \Rightarrow$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{x-\frac{\pi}{2}}{\cos x}.\lim_{x\to\frac{\pi}{2}} \sin x \Rightarrow \frac{0}{0} \ \text{ using L'Hopital's rule} \Rightarrow$$

$$= \lim_{x\to\frac{\pi}{2}} -\frac{1}{-\sin x}.\lim_{x\to\frac{\pi}{2}} \sin x = \frac{1}{\sin\frac{\pi}{2}}.\sin\frac{\pi}{2} = 1$$

4-2- <u>The slope of the curve</u> :

Secant to the curve is a line through two points on a curve.

Slopes and tangent lines :

1. we start with what we can calculate , namely the slope of secant through P and a point Q nearby on the curve .
2. we find the limiting value of the secant slope (if it exists) as Q approaches p along the curve .
3. we take this number to be the slope of the curve at P and define the tangent to the curve at P to be the line through p with this slope .

The derivative of the function f is the slope of the curve :

$$\text{the slope} = m = f'(x) = \frac{dy}{dx}$$

EX-2- Write an equation for the tangent line at $x = 3$ of the curve :

$$f(x) = \frac{1}{\sqrt{2x+3}}$$

Sol.-

$$m = f'(x) = -\frac{1}{\sqrt{(2x+3)^3}} \Rightarrow [m]_{x=3} = f'(3) = -\frac{1}{27}$$

$$f(3) = \frac{1}{\sqrt{2*3+3}} = \frac{1}{3}$$

The equation of the tangent line is :

$$y - \frac{1}{3} = -\frac{1}{27}(x-3) \Rightarrow 27y + x = 12$$

4-3- Velocity and acceleration and other rates of changes :

- The average velocity of a body moving along a line is :

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{f(t+\Delta t) - f(t)}{\Delta t} = \frac{displacement}{time\ travelled}$$

The instantaneous velocity of a body moving along a line is the derivative of its position $s = f(t)$ with respect to time t .

i.e. $$v = \frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t}$$

- The rate at which the particle's velocity increase is called its acceleration a . If a particle has an initial velocity v and a constant acceleration a, then its velocity after time t is $v + at$.

$$average\ acceleration = a_{av} = \frac{\Delta v}{\Delta t}$$

The acceleration at an instant is the limit of the average acceleration for an interval following that instant , as the interval tends to zero .

i.e. $$a = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t}$$

- The average rate of a change in a function $y = f(x)$ over the interval from x to $x + \Delta x$ is :

$$average\ rate\ of\ change = \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

The instantaneous rate of change of f at x is the derivative.

$$f'(x) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$ provided the limit exists .

EX-3- The position s (in meters) of a moving body as a function of time t (in second) is : $s = 2t^2 + 5t - 3$; find :

a) The displacement and average velocity for the time interval from $t = 0$ to $t = 2$ seconds .

b) The body's velocity at $t = 2$ seconds .

Sol.-

a) 1) $\Delta s = s(t+\Delta t) - s(t) = 2(t+\Delta t)^2 + 5(t+\Delta t) - 3 - \left[2t^2 + 5t - 3\right]$

$= (4t+5)\Delta t + 2(\Delta t)^2$

$at\ t=0\ and\ \Delta t = 2 \Rightarrow \Delta s = (4*0+5)*2+2*2^2 = 18$

2) $v_{av} = \dfrac{\Delta s}{\Delta t} = \dfrac{(4t+5)\Delta t + 2(\Delta t)^2}{\Delta t} = 4t+5+2.\Delta t$

$at\ t=0\ and\ \Delta t = 2 \Rightarrow v_{av} = 4*0+5+2*2 = 9$

b) $v(t) = \dfrac{d}{dt} f(t) = 4t+5$

$v(2) = 4*2+5 = 13$

EX-4- A particle moves along a straight line so that after t (seconds), its distance from O a fixed point on the line is s (meters), where $s = t^3 - 3t^2 + 2t$:

i) when is the particle at O ?
ii) what is its velocity and acceleration at these times ?
iii) what is its average velocity during the first second ?
iv) what is its average acceleration between $t = 0$ and $t = 2$?

Sol. –

i) $at\ s=0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-1)(t-2) = 0$

$either\ t=0\ or\ t=1\ or\ t=2\ sec.$

ii) $velocity = v(t) = 3t^2 - 6t + 2 \Rightarrow v(0) = 2m/s$

$\Rightarrow v(1) = -1m/s$

$\Rightarrow v(2) = 2m/s$

$acceleration = a(t) = 6t - 6 \Rightarrow a(0) = -6m/s^2$

$\Rightarrow a(1) = 0m/s^2$

$\Rightarrow a(2) = 6m/s^2$

iii) $v_{av} = \dfrac{\Delta s}{\Delta t} = \dfrac{s(1)-s(0)}{1-0} = \dfrac{1-3+2-0}{1} = 0m/s$

iv) $a_{av} = \dfrac{\Delta v}{\Delta t} = \dfrac{v(2)-v(0)}{2-0} = \dfrac{2-2}{2} = 0m/s^2$

4-4- Maxima and Minima :

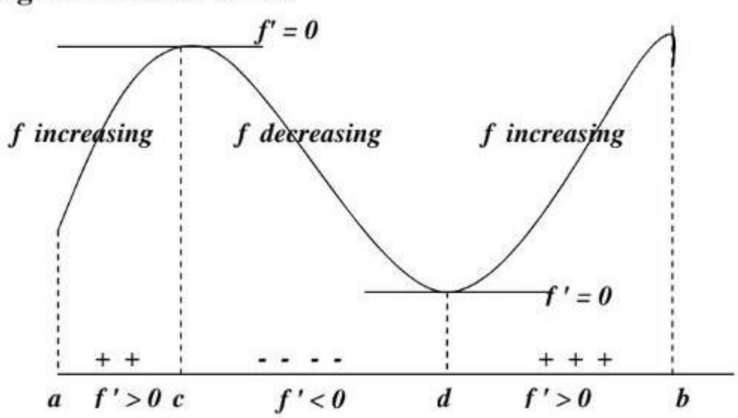
Increasing and decreasing function **: Let f be defined on an interval and x_1 , x_2 denoted a number on that interval :**

- **If $f(x_1) < f(x_2)$ when ever $x_1 < x_2$ then f is increasing on that interval .**
- **If $f(x_1) > f(x_2)$ when ever $x_1 < x_2$ then f is decreasing on that interval .**
- **If $f(x_1) = f(x_2)$ for all values of x_1 , x_2 then f is constant on that interval .**

The first derivative test for rise and fall **: Suppose that a function f has a derivative at every point x of an interval I. Then :**

- **f increases on I if $f'(x) > o, \quad \forall x \in I$**
- **f decreases on I if $f'(x) > o, \quad \forall x \in I$**

If f' changes from positive to negative values as x passes from left to right through a point c , then the value of f at c is a local maximum value of f , as shown in below figure . That is $f(c)$ is the largest value the function takes in the immediate neighborhood at $x = c$.



Similarly , if f' changes from negative to positive values as x passes left to right through a point d , then the value of f at d is a local minimum value of f . That is $f(d)$ is the smallest value of f takes in the immediate neighborhood of d .

EX-5 – **Graph the function :** $y = f(x) = \frac{x^3}{3} - 2x^2 + 3x + 2$.

Sol.- $f'(x) = x^2 - 4x + 3 \Rightarrow (x-1)(x-3) = 0 \Rightarrow x = 1,3$

Email (fatimatulzahraa.adnan.rahi@uomus.edu.iq)