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عنوان المحاضرة:

Vectors, Vectors in Space, Unit Vector, Scalar Product, Vector Product المتجهات، المتجهات في الفضاء، وحدة المتجه، ضرب القيمة العددية، ضرب المتجه

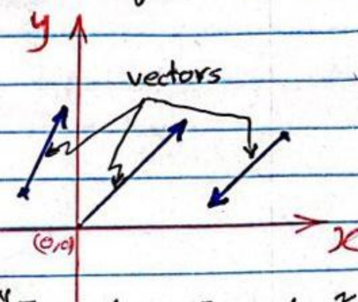
Vectors :- المتجهات

- The vector is only two pieces of information:
1- Direction
2- Length or Magnitude

المتجه عبارة عن مكونين وهما : اتجاه وطول أو كمية.

- We can graph a vector by an arrow that we can visualize on x-y plane and we can capture it by the arrow length and angle

- Vectors on graph could start from not just an origion, but from any where.



"Examples of Vectors"

Examples of Vectors :-

- To answer the question "What is the current temperature?" we use a single number (scalar); likewise the question about a mass ;
- While to answer the question "What is the current velocity of the wind?", we need more than just a single number. We need magnitude (speed) and direction. This where vectors come to handy.

Position, displacement, velocity, acceleration, force, momentum & torque are all physical quantities that can be represented mathmatically by vectors.

Vector Denoting :-

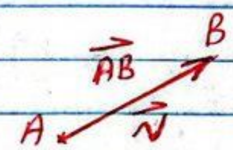
- Vectors are writing with arrow on top on equations.

Ex)

- Velocity vector ⟶ $\vec{v}$
- Force vector ⟶ $\vec{F}$

Note) Any variable symbol with no arrow on top means scalar.

- A vector can be geometrically represented by a direction line segment with a head & a tail ;



so vector $\vec{AB}$ is a vector from point A to B.

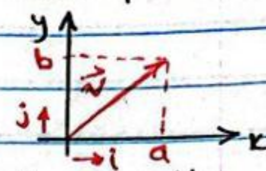
- Also, we can denote vector $\vec{AB}$ by a small case letter $\vec{v}$
- The length of the arrow A B Corresponds to the magnitude of the vector.
- The arrow points in the direction of the vector.

How to represent the vector mathematically :-

- Vector in plane :- كيف تمثل المتجه رياضياً

We can write vectors as columns. Let us take a very important special vector as example :-

$$i = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad j = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

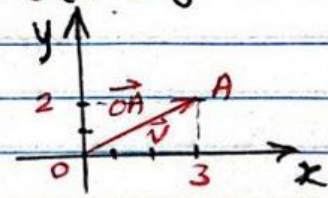


Any vector in the xy-plane can be written in terms of i & j using the triangle law & scalar multiplication.

$$\vec{v} = ai + bj = a\begin{pmatrix} 1 \\ 0 \end{pmatrix} + b\begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} a \\ b \end{pmatrix}$$

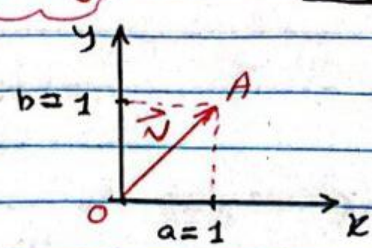
Ex)

$$\begin{pmatrix} 3 \\ 2 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 0 \end{pmatrix} + 2\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 3i + 2j$$



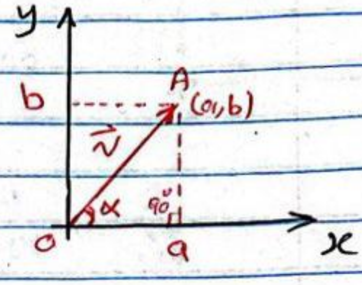
Notes

* If $a = b = 1$, then $\vec{v} = i + j$ is a "unit vector"



Finding the length/magnitude and the direction of vector :- إيجاد طول (مقدار) و اتجاه المتجه

If $\vec{v} = a\,i + b\,j$ ---① , then the length/magnitude of a vector $\vec{v}$ is :-



$$|v| = \sqrt{a^2 + b^2} \quad \text{---} ②$$

(نظرية فيثاغورس)

- It's a Pythagorean theorem

$$\left.\begin{aligned} a &= |v|\cos\alpha \\ b &= |v|\sin\alpha \end{aligned}\right\} \text{---} ③$$

$$\tan\alpha = \frac{b}{a}$$

Substitute eq. ③ in ① yields ;

$$\vec{v} = |v|(\cos\alpha\, i + \sin\alpha\, j)$$

$\vec{v}$ -- vector symbol
$|v|$ -- vector length
i, j -- unit vector components (basis/Fundamental vector components)
α -- vector angle with x-axis

Ex) Find a vector in plane of length (7 units) & makes angle (35°) with x-axis ?

Solution

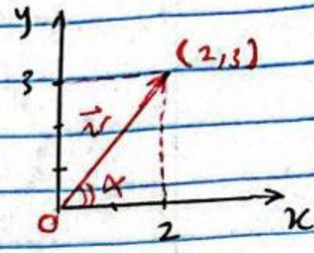
since $|\vec{v}| = 7$ & $\alpha = 35°$

$\therefore \vec{v} = 7 * (\cos 35\, i + \sin 35\, j)$

$\boxed{\vec{v} = 5.7\, i + 4\, j}$ Ans

Ex) Find the angle between the vector $\vec{v} = 2i + 3j$ and the x-axis?

Solution



$|\vec{v}| = \sqrt{a^2 + b^2}$; $a = 2$, $b = 3$

$|\vec{v}| = \sqrt{2^2 + 3^2} = \sqrt{13}$

$a = |v| \cos\alpha \Longrightarrow \cos\alpha = \frac{a}{|v|} = \frac{2}{\sqrt{13}}$

$\therefore \alpha = \cos^{-1} \frac{2}{\sqrt{13}} = \boxed{56.3^\circ}$ Ans

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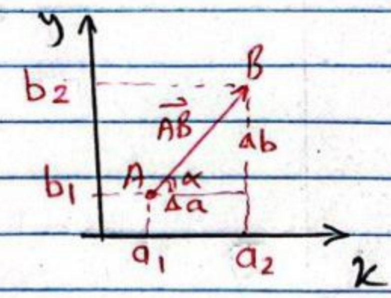
Vectors with tail not in origin :-

المتجهات بنقطة ابتداء ليس في (0,0)

Vectors can be start not from the origion, but from any where like A to B

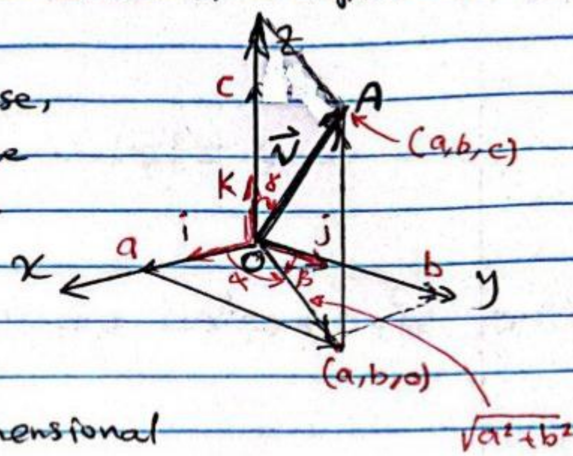


$\therefore \vec{AB} = \Delta a\, i + \Delta b\, j$

$\vec{AB} = (a_2 - a_1)\, i + (b_2 - b_1)\, j$

Vectors in a space :-

- This could be in three or higher dimensions



- Similar to the 2-D case, but we now have three basis vectors i, j & k.
- From these three components unit vectors we can describe any vector in three-dimensional space.

$$\vec{V} = \vec{OA} = a i + b j + c K$$

where :-

i, j, K = Basis or Fundamental unit vector.

a, b, c = Directional numbers (scalars).

α, β, γ = Directional angles.

$$|V| = |OA| = \sqrt{a^2 + b^2 + c^2}$$

$a = |V| * \cos\alpha$

$b = |V| * \cos\beta$

$c = |V| * \cos\gamma$

$$\frac{\vec{V}}{|V|} = \cos\alpha\, i + \cos\beta\, j + \cos\gamma\, K$$ = unit vector with direction of $\vec{V}$

And,

$$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1$$

Ex) Find a vector in space of length (5 units) that makes angles (70°) with x-axis, (85°) with y-axis ?

Solution

$\alpha = 70^\circ \quad , \beta = 85^\circ \quad , |v| = 5$

$\gamma = ? \quad , \vec{v} = ?$

$\cos^2\alpha + \cos^2\beta + \cos^2\gamma = 1 \Longrightarrow \cos^2 70 + \cos^2 85 + \cos^2\gamma = 1$

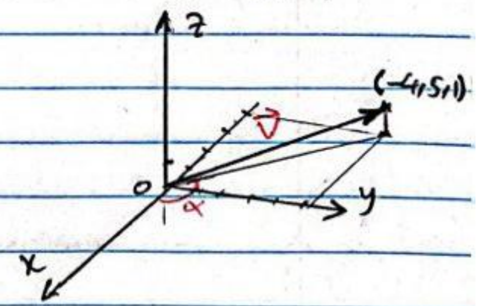
$\therefore \boxed{\cos\gamma = 0.935}$

$\therefore \vec{v} = |v|(\cos\alpha\, i + \cos\beta\, j + \cos\gamma\, k)$

$= 5(\cos 70\, i + \cos 85\, j + 0.935\, k)$

$\boxed{\vec{v} = 1.7\, i + 0.436\, j + 4.675\, k}$ Ans

Ex) Find the angle between the vector $\vec{v} = -4i + 5j + k$ and the x-axis?



Solution

$a = -4 \,, \; b = 5 \,, \; c = 1$

$|v| = \sqrt{a^2 + b^2 + c^2}$

$|v| = \sqrt{(-4)^2 + (5)^2 + (1)^2} = \sqrt{42}$

$\cos\alpha = \dfrac{a}{|v|} \Longrightarrow \alpha = \cos^{-1}\dfrac{a}{|v|} = \cos^{-1}\dfrac{-4}{\sqrt{42}}$

$\boxed{\alpha = 128^\circ}$

Scalar product :- (Dot product)

ضرب القيم العددية

Let $\vec{A} = a_1 i + a_2 j + a_3 k$

And $\vec{B} = b_1 i + b_2 j + b_3 k$

Then, $\vec{A} \cdot \vec{B} = |A|\,|B| \cos\theta$

where θ — is the angle between $\vec{A}$ & $\vec{B}$

Properties :-

1- $\vec{A} \cdot \vec{A} = |A|^2$

2- $\vec{A} \cdot \vec{B} = \vec{B} \cdot \vec{A} = a_1 b_1 + a_2 b_2 + a_3 b_3$

3- $\cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|A|\,|B|} = \dfrac{a_1 b_1 + a_2 b_2 + a_3 b_3}{\sqrt{a_1^2 + a_2^2 + a_3^2} + \sqrt{b_1^2 + b_2^2 + b_3^2}}$

4- $\vec{A} \perp \vec{B} \Rightarrow \vec{A} \cdot \vec{B} = 0$ [Orthogonal Vectors] متجهات متعامدة

5- $ai + bj \perp bi - aj$

EX) Find the angle "θ" between $\vec{A} = i - 2j - 2k$ & $\vec{B} = 6i + 3j + 2k$?

Solution

$\vec{A} \cdot \vec{B} = (1 * 6) + (-2 * 3) + (-2 * 2) = \boxed{-4}$

$|A| = \sqrt{1^2 + (-2)^2 + (-2)^2} = \sqrt{9} = \boxed{3}$

$|B| = \sqrt{6^2 + 3^2 + 2^2} = \sqrt{49} = \boxed{7}$

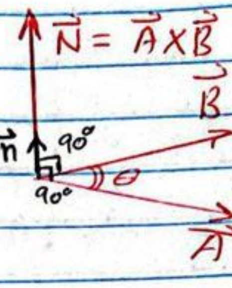
$\Rightarrow |A|\,|B| = \boxed{21}$

$\therefore \cos\theta = \dfrac{\vec{A} \cdot \vec{B}}{|A|\,|B|} \Rightarrow \theta = \cos^{-1} \dfrac{-4}{21} \approx \boxed{101^\circ}$ Ans

Vector Product :- (Cross-product)

ضرب المتجهات

Normal vector is what yields from vector product or cross product.



$$\vec{N} = \vec{A} \times \vec{B} = \vec{n}\,|A||B|\sin\theta$$

where $\vec{n}$ -- is a normal unit vector

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \quad ; \text{where, } \begin{array}{l} \vec{A} = a_1 i + a_2 j + a_3 k \\ \vec{B} = b_1 i + b_2 j + b_3 k \end{array}$$

Properties :-

1- $\vec{A} \times \vec{A} = 0 \longrightarrow$ " $\sin 0 = 0$ "

2- $\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}$

3- $\vec{A} \parallel \vec{B} \Longrightarrow \vec{A} \times \vec{B} = 0 \longrightarrow \sin 0 = 0$

4- Area of $\Delta ABC = \frac{1}{2}|\vec{A} \times \vec{B}|$

Ex) Find $\vec{A}\times\vec{B}$ & $\vec{B}\times\vec{A}$ if

$\vec{A}=2i+j+k$

$\vec{B}=-4i+3j+k$

Solution

$$\vec{A}\times\vec{B}=\begin{vmatrix} i & j & k \\ 2 & 1 & 1 \\ -4 & 3 & 1 \end{vmatrix}=\begin{vmatrix} 1 & 1 \\ 3 & 1 \end{vmatrix} i-\begin{vmatrix} 2 & 1 \\ -4 & 1 \end{vmatrix} j+\begin{vmatrix} 2 & 1 \\ -4 & 3 \end{vmatrix} k$$

$$=(1*1-(3*1))\,i-(2*1-(-4*1))\,j+(2*3-(-4*1))\,k$$

$$\boxed{\vec{A}\times\vec{B}=-2i-6j+10k}$$

but,

$$\vec{B}\times\vec{A}=-\vec{A}\times\vec{B}=\boxed{2i+6j-10k}$$

Triple product :- الضرب الثلاثي المحدود

A- Scalar triple product :-

$$\vec{A}\cdot(\vec{B}\times\vec{C})=(\vec{A}\times\vec{B})\cdot\vec{C}$$

Note)

1- Box volume is $\Longrightarrow$ $V_{box}=|\vec{A}\cdot\vec{B}\times\vec{C}|$

2- Pyramid volume is $\longrightarrow$ $V_{py}=\frac{1}{6}|\vec{A}\cdot\vec{B}\times\vec{C}|$

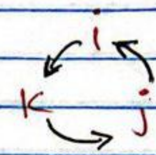
B- Vector triple product :-

$$\vec{A}\times(\vec{B}\times\vec{C})=(\vec{A}\cdot\vec{C})*B-(\vec{A}\cdot\vec{B})\vec{C}$$

Note)

$\hat{i}\cdot\hat{i}=\hat{j}\cdot\hat{j}=k\cdot k=1$; $i\times i=j\times j=k\times k=0$

$i \cdot j = j \cdot k = k \cdot i = 0$



$i \times j = k$
$j \times k = i$
$k \times i = j$

HW#2

1- Find the Length & direction of these vectors & the angles make with the x-axis?

a- $5i + 12j$
b- $\sqrt{3}\, i + j$

2- Find a vector 6 units long in the direction of $\vec{A} = 2i + 2j - k$

3- Find the area of the triangle whose vertices are $A(1,-1,0)$, $B(2,1,-1)$, $C(-1,1,2)$?

4- If $\vec{A} = 2i - j$ & $B = i + 3j - 2k$, find $A \times B$, then calculate $(A \times B) \cdot A$?

---- نهاية محاضرة " Vectors, Vectors in Space, Unit Vector, Scalar Product, Vector Product المتجهات، المتجهات في الفضاء، وحدة المتجه، ضرب القيمة العددية، ضرب المتجه " ----