



عنوان المحاضرة: Detremnants, Properties, Cramer's Rule, App of
Det. المحددات، الخصائص، قاعدة كرامر، تطبيقات المحددات

المحددات - Determinants :

square array of numbers or symbols and closed by two vertical bars.

$$D = \begin{vmatrix} a_{11} & a_{12} & a_{13} & \dots & a_{1n} \\ a_{21} & a_{22} & a_{23} & \dots & a_{2n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & a_{m3} & \dots & a_{mn} \end{vmatrix} = |a_{ij}|$$

a_{ij} = individual elements -

i = row

j = column

n = order of determinant

D = (always be) value of determinant -

Minor a_{ij}

Designated as M_{ij} of order $n-1$

$$a_{ij} = \begin{vmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \\ a_{41} & a_{42} & a_{43} & a_{44} \end{vmatrix} \Rightarrow M_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{14} \\ a_{31} & a_{32} & a_{34} \\ a_{41} & a_{42} & a_{44} \end{vmatrix}$$

ويُعرف الصفوف المتبقية بعد الحذف ² تسمى minor.



Cofactor of a_{ij} :

Designated by A_{ij} is defined by :-

$$A_{ij} = (-1)^{i+j} M_{ij}$$

Laplace Expansion Formula :

$$\text{Det } A = \sum_{j=1}^n a_{ij} A_{ij} \leftarrow (i = \text{constant, summing along row } i)$$

$$\text{Det } A = \sum_{i=1}^n a_{ij} A_{ij} \leftarrow (\text{summing along column } j)$$

Example

Find the Determinant of $D = \begin{vmatrix} 3 & 0 & -2 \\ 1 & 4 & 1 \\ 3 & -2 & 5 \end{vmatrix}$

Solution

→ Lets do it about first column, which means $j=1 = \text{constant}$

$$\therefore D = (3)(-1)^{1+1} \begin{vmatrix} 4 & 1 \\ -2 & 5 \end{vmatrix} + (1)(-1)^{1+2} \begin{vmatrix} 3 & -2 \\ 3 & 5 \end{vmatrix} + (3)(-1)^{1+3} \begin{vmatrix} 3 & 0 \\ 1 & 4 \end{vmatrix}$$

$$\therefore D = 94$$

→ Lets expansion it about 1st row, which means $i=1 = \text{constant}$

$$\therefore D = (3)(-1)^{1+1} \begin{vmatrix} 4 & 1 \\ -2 & 5 \end{vmatrix} + (0)(-1)^{1+2} \begin{vmatrix} 1 & 1 \\ 3 & 5 \end{vmatrix} + (-2)(-1)^{1+3} \begin{vmatrix} 1 & 4 \\ 3 & -2 \end{vmatrix}$$

$$= 94$$

ok

Matrices :-

المصفوفات

A rectangular array of numbers or symbols with each element being distinct and separate (m rows, n columns) is called an $m \times n$ matrix. When certain laws of combination, yet to be specified, are laid down,

Example :-

$$y_i = \sum_{j=1}^n a_{ij} x_j \quad \text{where } i=1, 2, \dots, m$$

Solution

$$y_1 = a_{11} x_1 + a_{12} x_2 + \dots + a_{1n} x_n$$

$$\vdots$$

$$y_m = a_{m1} x_1 + a_{m2} x_2 + \dots + a_{mn} x_n$$

This can be written as a matrix

$$\begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_m \end{bmatrix} = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

$$[Y] = [A] [X]$$

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Special Matrices :-

① Square matrix :-

$$[A] = \begin{bmatrix} a_{11} & \dots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \dots & a_{nn} \end{bmatrix} \Rightarrow (m=n)$$

② Diagonal matrix :-

$$[A] = \begin{bmatrix} a_{11} & 0 & \dots & 0 \\ 0 & a_{22} & & 0 \\ \vdots & \vdots & a_{33} & \\ 0 & 0 & 0 & a_{nn} \end{bmatrix}$$

③ Unit matrix :- It may called by Identity matrix

$$[I] = \begin{bmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & 1 & 0 & & 0 \\ 0 & 0 & 1 & & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & & 1 \end{bmatrix}$$

④ Upper triangular matrix :-

$$[U] = \begin{bmatrix} u_{11} & u_{12} & u_{13} & \dots & u_{1n} \\ 0 & u_{22} & u_{23} & \dots & u_{2n} \\ 0 & 0 & u_{33} & \dots & u_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & u_{nn} \end{bmatrix}$$

⑤ Upper unit triangular matrix :-

$$[I] = [U] \quad \text{with diagonal elements equal to } 1.$$



⑥ Lower triangular matrix ~

$[L]$ = all elements above the diagonal are zero

⑦ Inverse of Square matrix $[A]$ ~

Defined by ~

(Identity or unit matrix)

$$[A]^{-1} [A] = [I]$$

$$a^{-1} a = \frac{a}{a} = 1$$

where,

$[A]^{-1}$ is called A inverse

⑧ Tridiagonal matrix = - مصفوفة قطرية

$$\begin{bmatrix} a_1 & c_1 & & & \\ a_2 & b_2 & c_2 & & \\ & a_3 & b_3 & c_3 & \\ & & & \ddots & \\ & & & a_{n-1} & b_{n-1} & c_{n-1} \\ & & & & a_n & b_n \end{bmatrix}$$

Zero (circled) in the top-right and bottom-left corners.

⑨ Transpose of a matrix :-

A matrix Form when columns & rows are interchange, designated as A^T or $[A]^T$

$$[A] = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix} \Rightarrow [A]^T = \begin{bmatrix} a_{11} & a_{21} & \dots & a_{m1} \\ a_{12} & a_{22} & \dots & a_{m2} \\ \vdots & \vdots & \ddots & \vdots \\ a_{1n} & a_{2n} & \dots & a_{mn} \end{bmatrix}$$



⑩ Elementary matrix operations :-

a- Two matrices are equal

$$[A] = [B] \quad \text{if} \quad a_{ij} = b_{ij} \quad \begin{cases} i = 1, 2, \dots, m \\ j = 1, 2, \dots, n \end{cases}$$

b- Matrix addition :-

$$[A] + [B] = [C] \quad \text{if} \quad c_{ij} = a_{ij} + b_{ij}$$

c- Matrix multiplication :-

$$[C] = [A][B] \quad \text{if} \quad c_{ik} = \sum_{j=1}^n a_{ij} b_{jk}$$

$$\begin{cases} i = 1, 2, \dots, m \\ k = 1, 2, \dots, q \end{cases}$$

Example :-

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \\ b_{31} & b_{32} \end{bmatrix} = \begin{bmatrix} \underbrace{a_{11}b_{11} + a_{12}b_{21} + a_{13}b_{31}}_{C_{11}} & \underbrace{a_{11}b_{12} + a_{12}b_{22} + a_{13}b_{32}}_{C_{12}} \\ \underbrace{a_{21}b_{11} + a_{22}b_{21} + a_{23}b_{31}}_{C_{21}} & \underbrace{a_{21}b_{12} + a_{22}b_{22} + a_{23}b_{32}}_{C_{22}} \end{bmatrix}$$
$$= \begin{bmatrix} C_{11} & C_{12} \\ C_{21} & C_{22} \end{bmatrix}$$

Matrix Properties :-

1- Matrix multiplication is associative

$$[A][B][C] = ([A][B])[C]$$

2- Not Commutative

$$[A][B] \neq [B][A]$$

But there is exception $\Rightarrow [I][A] = [A][I] = [A]$

3- Is distributive $\Rightarrow [A][B] + [A][C] = [A][B + C]$



Properties of Determinant :

a- $\text{Det}[I] = 1$

b- $\text{Det}[A] = \text{Det}[A]^{-1}$

c- $(\text{Det}[A])^{-1} = \frac{1}{\text{Det}[A]}$

d- IF $[A]$ and $[B]$ are square matrix, then

$$\text{Det}([A][B]) = \text{Det}[A] \cdot \text{Det}[B]$$

e- IF $[A]$ has size $n \times n$ and K is a constant, then,

$$\text{Det}(K[A]) = K^n \cdot \text{Det}[A]$$

f- IF to the element of any row (or column) are added K times the corresponding elements of any other row (or column), the determinant is unchanged

Example

Find determinant $D = \begin{vmatrix} 3 & 1 & 3 \\ 0 & 2 & 4 \\ 1 & 2 & 3 \end{vmatrix} = -8$

and, with property "f":

$$D = \begin{vmatrix} 3 & 1 & (3 - (2 \times 1)) \\ 0 & 2 & (4 - (2 \times 2)) \\ 1 & 2 & (3 - (2 \times 2)) \end{vmatrix} = -8$$

g- IF any two rows (or columns) of $[A]$ are interchanged, yielding a new matrix $[B]$, then

$$\text{Det}[B] = -\text{Det}[A]$$



h- If $[A]$ is triangular, then $\text{Det}[A]$ is the product of the diagonal elements.

$$\text{Det}[A] = |A| = a_{11} \cdot a_{22} \cdot a_{33} \cdots a_{nn}$$

Example

Let $A = \begin{bmatrix} 0 & 2 & -1 \\ 4 & 3 & 5 \\ 2 & 0 & -4 \end{bmatrix}$

Find the $\text{Det}[A]$ with several different methods.

Solution

① By using "Laplace Expansion Formula":

$$\text{Det } A = \sum_{i=1}^n a_{ij} A_{ij} \quad ; \quad A_{ij} = (-1)^{i+j} M_{ij}$$

a- Using row #1

$$\text{Det } A = |A| = (0) \begin{vmatrix} 3 & 5 \\ 0 & -4 \end{vmatrix} - (2) \begin{vmatrix} 4 & 5 \\ 2 & -4 \end{vmatrix} + (-1) \begin{vmatrix} 4 & 3 \\ 2 & 0 \end{vmatrix}$$

$$= 0 - (2)[-16 - 10] + (-1)[0 - 6]$$

$$|A| = \boxed{58}$$

b- Using row #2:

$$\text{Det } A = |A| = -(4) \begin{vmatrix} 2 & -1 \\ 0 & -4 \end{vmatrix} + (3) \begin{vmatrix} 0 & -1 \\ 2 & -4 \end{vmatrix} - (5) \begin{vmatrix} 0 & 2 \\ 2 & 0 \end{vmatrix}$$

$$= -(4)(-8) + (3)(2) - (5)(-4)$$

$$|A| = \boxed{58}$$

c- Using property (g), then (f) to get "upper triangular matrix"

$$|A| = \begin{vmatrix} 0 & 2 & -1 \\ 4 & 3 & 5 \\ 2 & 0 & -4 \end{vmatrix} = - \begin{vmatrix} 2 & 0 & -4 \\ 4 & 3 & 5 \\ 0 & 2 & -1 \end{vmatrix}$$

$$= - \begin{vmatrix} 2 & 0 & -4 \\ 0 & 3 & 13 \\ 0 & 2 & -4 \end{vmatrix} \leftarrow \{a_{2j} - 2(a_{1j})\}$$



$$= - \begin{vmatrix} 2 & 0 & -4 \\ 0 & 3 & 13 \\ 0 & 0 & -\frac{29}{3} \end{vmatrix} \leftarrow \left[a_{3j} - \frac{2}{3}(a_{2j}) \right] \quad \left. \begin{array}{l} \text{Now we have} \\ \text{"upper triangular"} \\ \text{matrix"} \end{array} \right\}$$

$$= -(2 \times 3) \left(-\frac{29}{3} \right)$$

$$= \boxed{58}$$

i- IF all the elements of any row or column are zero, then $\text{Det } A = 0$

j- IF any one row (or column) a of A is separated as $a = b + c$, then

$$\text{Det } A|_a = \text{Det } A|_b + \text{Det } A|_c$$

Ex

$$\begin{vmatrix} 6+2 & -3+1 & 5+4 \\ 3 & 0 & 2 \\ 1 & -6 & 7 \end{vmatrix} = \begin{vmatrix} 6 & -3 & 5 \\ 3 & 0 & 2 \\ 1 & -6 & 7 \end{vmatrix} + \begin{vmatrix} 2 & 1 & 4 \\ 3 & 0 & 2 \\ 1 & -6 & 7 \end{vmatrix}$$

k- The determinant of A & its transpose are equal

$$\text{Det } A^T = \text{Det } A$$

l- In general $\text{Det}(A+B) \neq \text{Det } A + \text{Det } B$



Cramer's Rule → {The name is coming after}
Gabriel Cramer (1704 - 1752)}

$$x_i = \frac{\text{Det } A_i}{\text{Det } A} \quad ; \quad [A][X] = [B]$$

Where;

$\text{Det } A$ = Determinant of matrix $[A]$.

$\text{Det } A_i$ = Determinant of matrix $[A_i]$, where $[A_i]$ is matrix $[A]$ with its i -th column replaced by column matrix $[B]$.

Ex)

$$3x_1 - 3x_2 + 5x_3 = 4$$

$$x_1 + 2x_2 - 6x_3 = 3$$

$$2x_1 - x_2 + 3x_3 = 1$$

Find $x_1, x_2, \& x_3$

Solution

In the matrix form, we have :

$$\begin{bmatrix} 3 & -3 & 5 \\ 1 & 2 & -6 \\ 2 & -1 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}$$

$$[A] [X] = [B]$$

We can find $\text{Det } A = 20$

$$\text{Det } A_1 = \begin{vmatrix} 4 & -3 & 5 \\ 3 & 2 & -6 \\ 1 & -1 & 3 \end{vmatrix} = 20$$

$$\text{Similarly, } \text{Det } A_2 = \begin{vmatrix} 3 & 4 & 5 \\ 1 & 3 & -6 \\ 2 & 1 & 3 \end{vmatrix} = -40$$

$$\text{Det } A_3 = -20$$



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Then,

$$x_1 = \frac{20}{20} = 1, \quad x_2 = \frac{-40}{20} = -2, \quad x_3 = \frac{-20}{20} = -1$$

Application of the Determinant

* First we need to define the **adjugate** of (A) to be the transpose of the Cofactor matrix.

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \xrightarrow{\text{then}} \text{adj}(A) = \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = (A_{ij})^T$$

(Cofactor Transpose)

Inverse Matrix

$$A^{-1} = \frac{\text{adj}(A)}{\text{Det}(A)}, \quad \text{Det}(A) \neq 0$$

Ex) Find the inverse of the matrix

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \text{ using the adjugate theorem.}$$

Solution

First, we will find the determinant of the matrix

$$\text{Det } A = (1) \begin{vmatrix} 0 & 1 \\ 2 & 1 \end{vmatrix} - 2 \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} + 3 \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix}$$

$$= -2 - 2(3-1) + 3(6) = -2 - 4 + 18$$

$$\text{Det } A = 12$$



Now, we need to Find $\text{adj}(A)$. To do so,
1st we will find the cofactor matrix of (A) . This is
given by

$$\text{cof}(A) = \begin{bmatrix} + \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} & - \begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} & + \begin{vmatrix} 3 & 0 \\ 1 & 2 \end{vmatrix} \\ + \begin{vmatrix} 2 & 3 \\ 2 & 1 \end{vmatrix} & + \begin{vmatrix} 1 & 3 \\ 1 & 1 \end{vmatrix} & - \begin{vmatrix} 1 & 2 \\ 1 & 2 \end{vmatrix} \\ + \begin{vmatrix} 2 & 3 \\ 0 & 1 \end{vmatrix} & - \begin{vmatrix} 1 & 3 \\ 3 & 1 \end{vmatrix} & + \begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} \end{bmatrix} = \begin{bmatrix} -2 & -2 & 6 \\ 4 & -2 & 0 \\ 2 & 8 & -6 \end{bmatrix} = \text{adj}(A)$$

$$\therefore \text{adj}(A) = A_{ij}^T = \begin{bmatrix} -2 & 4 & 2 \\ -2 & -2 & 8 \\ 6 & 0 & -6 \end{bmatrix}$$

$$\therefore A^{-1} = \frac{1}{12} \begin{bmatrix} -2 & 4 & 2 \\ -2 & -2 & 8 \\ 6 & 0 & -6 \end{bmatrix} = \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix}$$

To verify our answer for (A^{-1}) . Compute
the product $A \cdot A^{-1}$ or $A^{-1} \cdot A$ & make sure
each product is equal to I (identity matrix)

$$A \cdot A^{-1} = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 0 & 1 \\ 1 & 2 & 1 \end{bmatrix} \cdot \begin{bmatrix} -\frac{1}{6} & \frac{1}{3} & \frac{1}{6} \\ -\frac{1}{6} & -\frac{1}{6} & \frac{2}{3} \\ \frac{1}{2} & 0 & -\frac{1}{2} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I$$



H.W

Find the inverse of the matrix

$$A = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{2} \\ -\frac{1}{6} & \frac{1}{3} & -\frac{1}{2} \\ -\frac{5}{6} & \frac{2}{3} & -\frac{1}{2} \end{bmatrix}$$

Using the "adjugate theorem"

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