8

Techniques of Integration

Over the next few sections we examine some techniques that are frequently successful when seeking antiderivatives of functions. Sometimes this is a simple problem, since it will be apparent that the function you wish to integrate is a derivative in some straightforward way. For example, faced with

$$\int x^{10}\,dx$$

we realize immediately that the derivative of x^{11} will supply an x^{10}: $(x^{11})' = 11x^{10}$. We don't want the "11", but constants are easy to alter, because differentiation "ignores" them in certain circumstances, so

$$\frac{d}{dx}\frac{1}{11}x^{11} = \frac{1}{11}11x^{10} = x^{10}.$$

From our knowledge of derivatives, we can immediately write down a number of antiderivatives. Here is a list of those most often used:

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} + C, \quad \text{if } n \neq -1$$

$$\int x^{-1}\,dx = \ln|x| + C$$

$$\int e^x\,dx = e^x + C$$

$$\int \sin x\,dx = -\cos x + C$$

$$\int \cos x\,dx = \sin x + C$$
$$\int \sec^2 x\,dx = \tan x + C$$
$$\int \sec x \tan x\,dx = \sec x + C$$
$$\int \frac{1}{1+x^2}\,dx = \arctan x + C$$
$$\int \frac{1}{\sqrt{1-x^2}}\,dx = \arcsin x + C$$

8.1 Substitution

Needless to say, most problems we encounter will not be so simple. Here's a slightly more complicated example: find

$$\int 2x\cos(x^2)\,dx.$$

This is not a "simple" derivative, but a little thought reveals that it must have come from an application of the chain rule. Multiplied on the "outside" is $2x$, which is the derivative of the "inside" function x^2. Checking:

$$\frac{d}{dx}\sin(x^2) = \cos(x^2)\frac{d}{dx}x^2 = 2x\cos(x^2),$$

so

$$\int 2x\cos(x^2)\,dx = \sin(x^2) + C.$$

Even when the chain rule has "produced" a certain derivative, it is not always easy to see. Consider this problem:

$$\int x^3\sqrt{1-x^2}\,dx.$$

There are two factors in this expression, x^3 and $\sqrt{1-x^2}$, but it is not apparent that the chain rule is involved. Some clever rearrangement reveals that it is:

$$\int x^3\sqrt{1-x^2}\,dx = \int (-2x)\left(-\frac{1}{2}\right)(1-(1-x^2))\sqrt{1-x^2}\,dx.$$

This looks messy, but we do now have something that looks like the result of the chain rule: the function $1-x^2$ has been substituted into $-(1/2)(1-x)\sqrt{x}$, and the derivative of $1-x^2$, $-2x$, multiplied on the outside. If we can find a function $F(x)$ whose derivative is $-(1/2)(1-x)\sqrt{x}$ we'll be done, since then

$$\begin{aligned}\frac{d}{dx}F(1-x^2) &= -2xF'(1-x^2) = (-2x)\left(-\frac{1}{2}\right)(1-(1-x^2))\sqrt{1-x^2}\\ &= x^3\sqrt{1-x^2}\end{aligned}$$

But this isn't hard:

$$\begin{aligned}\int -\frac{1}{2}(1-x)\sqrt{x}\,dx &= \int -\frac{1}{2}(x^{1/2}-x^{3/2})\,dx \qquad (8.1.1)\\ &= -\frac{1}{2}\left(\frac{2}{3}x^{3/2}-\frac{2}{5}x^{5/2}\right)+C\\ &= \left(\frac{1}{5}x-\frac{1}{3}\right)x^{3/2}+C.\end{aligned}$$

So finally we have

$$\int x^3\sqrt{1-x^2}\,dx = \left(\frac{1}{5}(1-x^2)-\frac{1}{3}\right)(1-x^2)^{3/2}+C.$$

So we succeeded, but it required a clever first step, rewriting the original function so that it looked like the result of using the chain rule. Fortunately, there is a technique that makes such problems simpler, without requiring cleverness to rewrite a function in just the right way. It sometimes does not work, or may require more than one attempt, but the idea is simple: guess at the most likely candidate for the "inside function", then do some algebra to see what this requires the rest of the function to look like.

One frequently good guess is any complicated expression inside a square root, so we start by trying $u = 1-x^2$, using a new variable, u, for convenience in the manipulations that follow. Now we know that the chain rule will multiply by the derivative of this inner function:

$$\frac{du}{dx} = -2x,$$

so we need to rewrite the original function to include this:

$$\int x^3\sqrt{1-x^2} = \int x^3\sqrt{u}\frac{-2x}{-2x}\,dx = \int \frac{x^2}{-2}\sqrt{u}\frac{du}{dx}\,dx.$$

Recall that one benefit of the Leibniz notation is that it often turns out that what looks like ordinary arithmetic gives the correct answer, even if something more complicated is going on. For example, in Leibniz notation the chain rule is

$$\frac{dy}{dx} = \frac{dy}{dt}\frac{dt}{dx}.$$

The same is true of our current expression:

$$\int \frac{x^2}{-2}\sqrt{u}\,\frac{du}{dx}\,dx = \int \frac{x^2}{-2}\sqrt{u}\,du.$$

Now we're almost there: since $u = 1 - x^2$, $x^2 = 1 - u$ and the integral is

$$\int -\frac{1}{2}(1-u)\sqrt{u}\,du.$$

It's no coincidence that this is exactly the integral we computed in (8.1.1), we have simply renamed the variable u to make the calculations less confusing. Just as before:

$$\int -\frac{1}{2}(1-u)\sqrt{u}\,du = \left(\frac{1}{5}u - \frac{1}{3}\right)u^{3/2} + C.$$

Then since $u = 1 - x^2$:

$$\int x^3\sqrt{1-x^2}\,dx = \left(\frac{1}{5}(1-x^2) - \frac{1}{3}\right)(1-x^2)^{3/2} + C.$$

To summarize: if we suspect that a given function is the derivative of another via the chain rule, we let u denote a likely candidate for the inner function, then translate the given function so that it is written entirely in terms of u, with no x remaining in the expression. If we can integrate this new function of u, then the antiderivative of the original function is obtained by replacing u by the equivalent expression in x.

Even in simple cases you may prefer to use this mechanical procedure, since it often helps to avoid silly mistakes. For example, consider again this simple problem:

$$\int 2x\cos(x^2)\,dx.$$

Let $u = x^2$, then $du/dx = 2x$ or $du = 2x\,dx$. Since we have exactly $2x\,dx$ in the original integral, we can replace it by du:

$$\int 2x\cos(x^2)\,dx = \int \cos u\,du = \sin u + C = \sin(x^2) + C.$$

This is not the only way to do the algebra, and typically there are many paths to the correct answer. Another possibility, for example, is: Since $du/dx = 2x$, $dx = du/2x$, and then the integral becomes

$$\int 2x\cos(x^2)\,dx = \int 2x\cos u\,\frac{du}{2x} = \int \cos u\,du.$$

The important thing to remember is that you must eliminate all instances of the original variable x.

EXAMPLE 8.1.1 Evaluate $\int (ax+b)^n\,dx$, assuming that a and b are constants, $a \neq 0$, and n is a positive integer. We let $u = ax + b$ so $du = a\,dx$ or $dx = du/a$. Then

$$\int (ax+b)^n\,dx = \int \frac{1}{a}u^n\,du = \frac{1}{a(n+1)}u^{n+1} + C = \frac{1}{a(n+1)}(ax+b)^{n+1} + C.$$

□

EXAMPLE 8.1.2 Evaluate $\int \sin(ax+b)\,dx$, assuming that a and b are constants and $a \neq 0$. Again we let $u = ax + b$ so $du = a\,dx$ or $dx = du/a$. Then

$$\int \sin(ax+b)\,dx = \int \frac{1}{a}\sin u\,du = \frac{1}{a}(-\cos u) + C = -\frac{1}{a}\cos(ax+b) + C.$$

□

EXAMPLE 8.1.3 Evaluate $\int_2^4 x\sin(x^2)\,dx$. First we compute the antiderivative, then evaluate the definite integral. Let $u = x^2$ so $du = 2x\,dx$ or $x\,dx = du/2$. Then

$$\int x\sin(x^2)\,dx = \int \frac{1}{2}\sin u\,du = \frac{1}{2}(-\cos u) + C = -\frac{1}{2}\cos(x^2) + C.$$

Now

$$\int_2^4 x\sin(x^2)\,dx = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

A somewhat neater alternative to this method is to change the original limits to match the variable u. Since $u = x^2$, when $x = 2$, $u = 4$, and when $x = 4$, $u = 16$. So we can do this:

$$\int_2^4 x\sin(x^2)\,dx = \int_4^{16} \frac{1}{2}\sin u\,du = -\frac{1}{2}(\cos u)\Big|_4^{16} = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

An incorrect, and dangerous, alternative is something like this:

$$\int_2^4 x\sin(x^2)\,dx = \int_2^4 \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos(u)\Big|_2^4 = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{1}{2}\cos(16) + \frac{1}{2}\cos(4).$$

This is incorrect because $\int_2^4 \frac{1}{2}\sin u\,du$ means that u takes on values between 2 and 4, which is wrong. It is dangerous, because it is very easy to get to the point $-\frac{1}{2}\cos(u)\Big|_2^4$ and forget to substitute x^2 back in for u, thus getting the incorrect answer $-\frac{1}{2}\cos(4)+\frac{1}{2}\cos(2)$. A somewhat clumsy, but acceptable, alternative is something like this:

$$\int_2^4 x\sin(x^2)\,dx = \int_{x=2}^{x=4} \frac{1}{2}\sin u\,du = -\frac{1}{2}\cos(u)\Big|_{x=2}^{x=4} = -\frac{1}{2}\cos(x^2)\Big|_2^4 = -\frac{\cos(16)}{2}+\frac{\cos(4)}{2}.$$

□

EXAMPLE 8.1.4 Evaluate $\int_{1/4}^{1/2} \frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt$. Let $u=\sin(\pi t)$ so $du=\pi\cos(\pi t)\,dt$ or $du/\pi=\cos(\pi t)\,dt$. We change the limits to $\sin(\pi/4)=\sqrt{2}/2$ and $\sin(\pi/2)=1$. Then

$$\int_{1/4}^{1/2} \frac{\cos(\pi t)}{\sin^2(\pi t)}\,dt = \int_{\sqrt{2}/2}^{1} \frac{1}{\pi}\frac{1}{u^2}\,du = \int_{\sqrt{2}/2}^{1} \frac{1}{\pi}u^{-2}\,du = \frac{1}{\pi}\frac{u^{-1}}{-1}\Big|_{\sqrt{2}/2}^{1} = -\frac{1}{\pi}+\frac{\sqrt{2}}{\pi}.$$

□

Exercises 8.1.

Find the antiderivatives or evaluate the definite integral in each problem.

1. $\int (1-t)^9\,dt \Rightarrow$

2. $\int (x^2+1)^2\,dx \Rightarrow$

3. $\int x(x^2+1)^{100}\,dx \Rightarrow$

4. $\int \frac{1}{\sqrt[3]{1-5t}}\,dt \Rightarrow$

5. $\int \sin^3 x\cos x\,dx \Rightarrow$

6. $\int x\sqrt{100-x^2}\,dx \Rightarrow$

7. $\int \frac{x^2}{\sqrt{1-x^3}}\,dx \Rightarrow$

8. $\int \cos(\pi t)\cos\big(\sin(\pi t)\big)\,dt \Rightarrow$

9. $\int \frac{\sin x}{\cos^3 x}\,dx \Rightarrow$

10. $\int \tan x\,dx \Rightarrow$

11. $\int_0^{\pi} \sin^5(3x)\cos(3x)\,dx \Rightarrow$

12. $\int \sec^2 x\tan x\,dx \Rightarrow$

13. $\int_0^{\sqrt{\pi}/2} x\sec^2(x^2)\tan(x^2)\,dx \Rightarrow$

14. $\int \frac{\sin(\tan x)}{\cos^2 x}\,dx \Rightarrow$

15. $\int_3^4 \frac{1}{(3x-7)^2}\,dx \Rightarrow$

16. $\int_0^{\pi/6} (\cos^2 x-\sin^2 x)\,dx \Rightarrow$

17. $\int \frac{6x}{(x^2-7)^{1/9}}\,dx \Rightarrow$

18. $\int_{-1}^{1} (2x^3-1)(x^4-2x)^6\,dx \Rightarrow$

19. $\int_{-1}^{1} \sin^7 x\,dx \Rightarrow$

20. $\int f(x)f'(x)\,dx \Rightarrow$

8.2 Powers of sine and cosine

Functions consisting of products of the sine and cosine can be integrated by using substitution and trigonometric identities. These can sometimes be tedious, but the technique is straightforward. Some examples will suffice to explain the approach.

EXAMPLE 8.2.1 Evaluate $\int \sin^5 x\,dx$. Rewrite the function:

$$\int \sin^5 x\,dx = \int \sin x \sin^4 x\,dx = \int \sin x(\sin^2 x)^2\,dx = \int \sin x(1-\cos^2 x)^2\,dx.$$

Now use $u = \cos x$, $du = -\sin x\,dx$:

$$\begin{aligned}\int \sin x(1-\cos^2 x)^2\,dx &= \int -(1-u^2)^2\,du \\ &= \int -(1-2u^2+u^4)\,du \\ &= -u + \frac{2}{3}u^3 - \frac{1}{5}u^5 + C \\ &= -\cos x + \frac{2}{3}\cos^3 x - \frac{1}{5}\cos^5 x + C.\end{aligned}$$

□

EXAMPLE 8.2.2 Evaluate $\int \sin^6 x\,dx$. Use $\sin^2 x = (1-\cos(2x))/2$ to rewrite the function:

$$\begin{aligned}\int \sin^6 x\,dx = \int (\sin^2 x)^3\,dx &= \int \frac{(1-\cos 2x)^3}{8}\,dx \\ &= \frac{1}{8}\int 1 - 3\cos 2x + 3\cos^2 2x - \cos^3 2x\,dx.\end{aligned}$$

Now we have four integrals to evaluate:

$$\int 1\,dx = x$$

and

$$\int -3\cos 2x\,dx = -\frac{3}{2}\sin 2x$$

are easy. The $\cos^3 2x$ integral is like the previous example:

$$\begin{aligned}\int -\cos^3 2x\,dx &= \int -\cos 2x \cos^2 2x\,dx \\ &= \int -\cos 2x(1-\sin^2 2x)\,dx \\ &= \int -\frac{1}{2}(1-u^2)\,du \\ &= -\frac{1}{2}\left(u - \frac{u^3}{3}\right) \\ &= -\frac{1}{2}\left(\sin 2x - \frac{\sin^3 2x}{3}\right).\end{aligned}$$

And finally we use another trigonometric identity, $\cos^2 x = (1+\cos(2x))/2$:

$$\int 3\cos^2 2x\,dx = 3\int \frac{1+\cos 4x}{2}\,dx = \frac{3}{2}\left(x + \frac{\sin 4x}{4}\right).$$

So at long last we get

$$\int \sin^6 x\,dx = \frac{x}{8} - \frac{3}{16}\sin 2x - \frac{1}{16}\left(\sin 2x - \frac{\sin^3 2x}{3}\right) + \frac{3}{16}\left(x + \frac{\sin 4x}{4}\right) + C.$$

□

EXAMPLE 8.2.3 Evaluate $\int \sin^2 x \cos^2 x\,dx$. Use the formulas $\sin^2 x = (1-\cos(2x))/2$ and $\cos^2 x = (1+\cos(2x))/2$ to get:

$$\int \sin^2 x \cos^2 x\,dx = \int \frac{1-\cos(2x)}{2} \cdot \frac{1+\cos(2x)}{2}\,dx.$$

The remainder is left as an exercise. □

Exercises 8.2.

Find the antiderivatives.

1. $\int \sin^2 x\,dx \Rightarrow$

2. $\int \sin^3 x\,dx \Rightarrow$

3. $\int \sin^4 x\,dx \Rightarrow$

4. $\int \cos^2 x \sin^3 x\,dx \Rightarrow$

5. $\int \cos^3 x\,dx \Rightarrow$

6. $\int \sin^2 x \cos^2 x\,dx \Rightarrow$

7. $\int \cos^3 x \sin^2 x\,dx \Rightarrow$

8. $\int \sin x(\cos x)^{3/2}\,dx \Rightarrow$

9. $\int \sec^2 x \csc^2 x\,dx \Rightarrow$

10. $\int \tan^3 x \sec x\,dx \Rightarrow$