Stability Analysis

A system is said to be stable, if its output is under control. Otherwise, it is said to be unstable. A stable system produces a bounded output for a given bounded input.

Routh's Stability Criterion

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Table. I Routh table

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$	$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$	b_3	b_4	$\cdots$
s^{n-3}	$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$	$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^0	a_0				

$$\begin{array}{lllllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_0 & & & &
\end{array}$$

$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$	$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$	$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$
$b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$	$c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$	$d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$
$b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$	$c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$	

Criterion

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

Table. I Routh table

s^n	a_n	a_{n-2}	a_{n-4}	a_{n-6}	$\cdots$
s^{n-1}	a_{n-1}	a_{n-3}	a_{n-5}	a_{n-7}	$\cdots$
s^{n-2}	$b_1 = \frac{a_{n-1}a_{n-2} - a_n a_{n-3}}{a_{n-1}}$	$b_2 = \frac{a_{n-1}a_{n-4} - a_n a_{n-5}}{a_{n-1}}$	b_3	b_4	$\cdots$
s^{n-3}	$c_1 = \frac{b_1 a_{n-3} - a_{n-1} b_2}{b_1}$	$c_2 = \frac{b_1 a_{n-5} - a_{n-1} b_3}{b_1}$	c_3	c_4	$\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
s^0	a_0				

$$\begin{array}{llllll}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\vdots & \vdots & \vdots & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_0 & & & &
\end{array}$$

$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1}$ $b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1}$ $b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1}$	$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1}$ $c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1}$ $c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1}$	$d_1 = \frac{c_1 b_2 - b_1 c_2}{c_1}$ $d_2 = \frac{c_1 b_3 - b_1 c_3}{c_1}$

Example 1: The characteristic equation of a system is given below. Determine the stability of the system.

$$s^4 + 4s + 16s^2 + 10s^2 + 6 = 0$$

Solution:

Applying Routh Hurwitz Criteria and forming Routh array, we get

$$\begin{array}{llll}
s^4 & 1 & 16 & 5 \\
s^3 & 4 & 10 & 0\ (for\ missing\ term) \\
s^2 & \frac{16\times4-10\times1}{4} = 13.5 & \frac{5\times4-0\times1}{4} = 5 & \\
s^1 & \frac{13.5\times10-5\times4}{13.5} = 8.52 & 0 & \\
s^0 & 5 & &
\end{array}$$

In the Routh array formed, if we see in first column; all the elements are positive. There is no sign change. Hence the system in question is stable.

Example 2: The characteristic equation of a system is given below. Determine the stability of the system

$$4s^4 + 8s^3 + 2s^2 + 10s + 3 = 0$$

Solution:

Applying Routh Hurwitz Criteria and forming Routh array. We get,

$$\begin{array}{llll}
s^4 & 4 & 2 & 3 \\
s^3 & 8 & 10 & 0 \\
s^2 & \frac{8\times2-4\times10}{8} = -3 & \frac{8\times3-0\times4}{8} = 3 & \\
s^1 & \frac{-3\times10-3\times8}{-3} = 18 & & \\
s^0 & 3 & &
\end{array}$$

In the first column of the Routh Array formed above, there is one negative element. Also, there are two sign changes in first column. First is from 8 to -3 and second is from -3 to 18. Hence the system is questions is unstable and out of 4 poles, 2 are in the right half of s-plane.

Ex.: Consider the following polynomial, $S^4 + 2S^3 + 3S^2 + 4S + 5 = 0$, determine the stability of the system.

$$\begin{array}{llll||lllll}
s^4 & 1 & 3 & 5 & s^4 & 1 & 3 & 5 & \\
s^3 & 2 & 4 & 0 & s^3 & \not{2} & \not{4} & \not{0} & \text{Divide this row by two to get} \\
 & & & & & 1 & 2 & 0 & \\
s^2 & 1 & 5 & & s^2 & 1 & 5 & & \\
s^1 & -6 & & & s^1 & -3 & & & \\
s^0 & 5 & & & s^0 & 5 & & &
\end{array}$$

In this example, the sign changes twice in the first column so the polynomial equation $A(s) = 0$ has two roots with positive real parts.

Email: hasan.hamad.ali@uomus.edu.iq

Example:

Suppose that unity feedback is to be applied around the listed open-loop systems. Use Routh's stability criterion to determine whether the resulting closed-loop systems will be stable.

(a) $KG(s) = \frac{4(s+2)}{s(s^3+2s^2+3s+4)}$

(b) $KG(s) = \frac{2(s+4)}{s^2(s+1)}$

(c) $KG(s) = \frac{4(s^3+2s^2+s+1)}{s^2(s^3+2s^2-s-1)}$

Solution:

(a)

$$1 + KG = s^4 + 2s^3 + 3s^2 + 8s + 8 = 0.$$

The Rouh array is,

$$\begin{array}{llll} s^4 & : & 1 & 3 & 8 \\ s^3 & : & 2 & 8 \\ s^2 & : & b_1 & b_2 \\ s^1 & : & c_1 \\ s^0 & : & d_1 \end{array}$$

where

$$\begin{aligned} b_1 &= \frac{2 \times 3 - 8 \times 1}{2} = -1 \qquad b_2 = \frac{2 \times 8 - 1 \times 0}{2} = 8, \\ c_1 &= \frac{3a - 2b}{a} = \frac{-8 - 16}{-1} = 24, \qquad c_2 = 0 \\ d_1 &= b = 8. \end{aligned}$$

2 sign changes in the first column $\Longrightarrow$ 2 roots not in the LHP$\Longrightarrow$ unstable.

(b)

$$1 + KG = s^3 + s^2 + 2s + 8 = 0.$$

The Routh's array is,

$$\begin{array}{llll} s^3 & : & 1 & 2 \\ s^2 & : & 1 & 8 \\ s^1 & : & -6 \\ s^0 & : & 8 \end{array}$$

There are two sign changes in the first column of the Routh array. Therefore, there are two roots not in the LHP.

(c)

$$1 + KG = s^5 + 2s^4 + 3s^3 + 7s^2 + 4s + 4 = 0.$$

The Routh array is,

s^5	:	1	3	4
s^4	:	2	7	4
s^3	:	b_1	b_2	
s^2	:	c_1	c_2	
s^1	:	d_1		
s^0	:	e_1		

where

$$b_1 = \frac{6-7}{2} = \frac{-1}{2} \qquad b_2 = \frac{8-4}{2} = 2$$

$$c_1 = \frac{-7/2-4}{-1/2} = 15 \qquad c_2 = \frac{-4/2-0}{-1/2} = 4$$

$$d_1 = \frac{30+2}{15} = \frac{32}{15}$$

$$e_1 = 4$$

2 sign changes in the first column $\Longrightarrow$ 2 roots not in the LHP $\Longrightarrow$ unstable.

Example: Determining Acceptable Gain Values

So far we have discussed only one possible application of the Routh criterion, namely determining the number of roots with nonnegative real parts. In fact, it can be used to determine limits on design parameters, as shown below.

Consider a system whose closed-loop transfer function is

$$H(s) = \frac{K}{s(s^2+s+1)(s+2)+K}.$$

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s^4 + K = 0.$$

The Routh array is

s^4	1	3	K
s^3	3	2	0
s^2	$7/3$	K	
s^1	$2 - 9K/7$		
s^0	K		

so the s^1 row yields the condition that, for stability,

$$14/9 > K > 0.$$

Email: hasan.hamad.ali@uomus.edu.iq