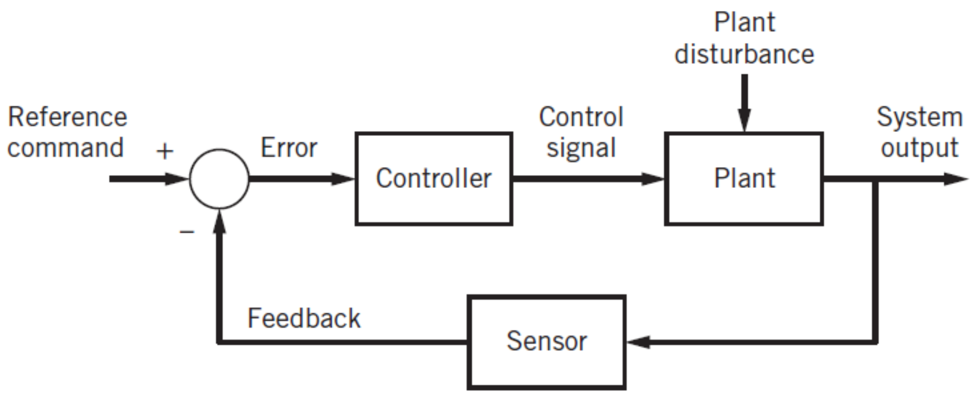
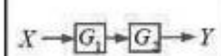
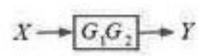
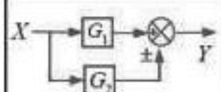
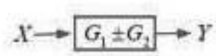
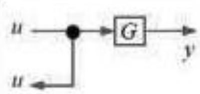
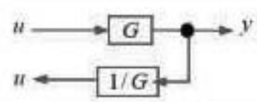
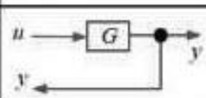
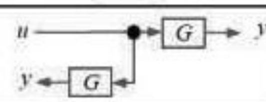
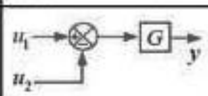
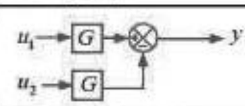
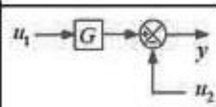
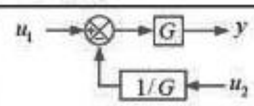
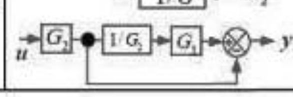
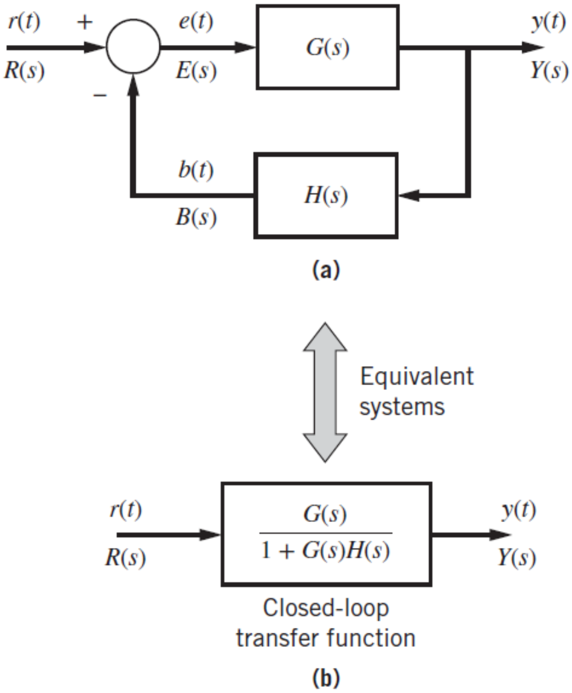
Time Response Analysis



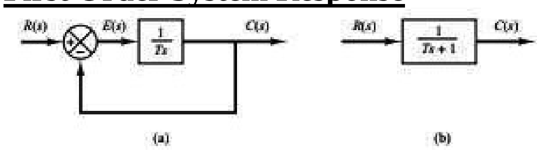
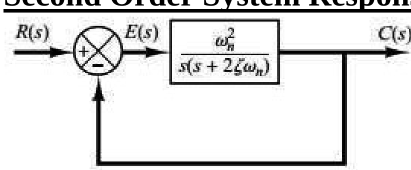
General closed-loop feedback control system.

Table 2: Basic rules with block diagram transformation

	Manipulation	Original Block Diagram	Equivalent Block Diagram	Equation
1	Combining Blocks in Cascade			$Y=(G_1G_2)X$
2	Combining Blocks in Parallel; or Eliminating a Forward Loop			$Y=(G_1 \pm G_2)X$
3	Moving a pickoff point behind a block			$y = Gu$ $u = \frac{1}{G}y$
4	Moving a pickoff point ahead of a block			$y=Gu$
5	Moving a summing point behind a block			$e_2=G(u_1-u_2)$
6	Moving a summing point ahead of a block		 	$y=Gu_1-u_2$ $y=(G_1-G_2)u$



Closed-loop systems: (a) system with forward and feedback paths and (b) equivalent closed-loop system.

First Order System Response	Second Order System Response
	 $$\frac{C(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Poles and Zeros of Laplace Transforms

In general, a Laplace transform can be expressed using the form

$$F(s) = \frac{a(s+z_1)(s+z_2)\cdots(s+z_m)}{(s+p_1)(s+p_2)\cdots(s+p_n)}$$

where a is a constant. The values $s = -z_1, s = -z_2, \ldots, s = -z_m$ that make $F(s) = 0$ are called the *zeros* of the transform $F(s)$ while the values $s = -p_1, s = -p_2, \ldots, s = -p_n$ that make $F(s) = \infty$ (or, the denominator of $F(s)$ equals zero) are called the *poles* of $F(s)$.

- **The control system is stable if all the poles of the closed loop transfer function present in the left half of the s plane.**

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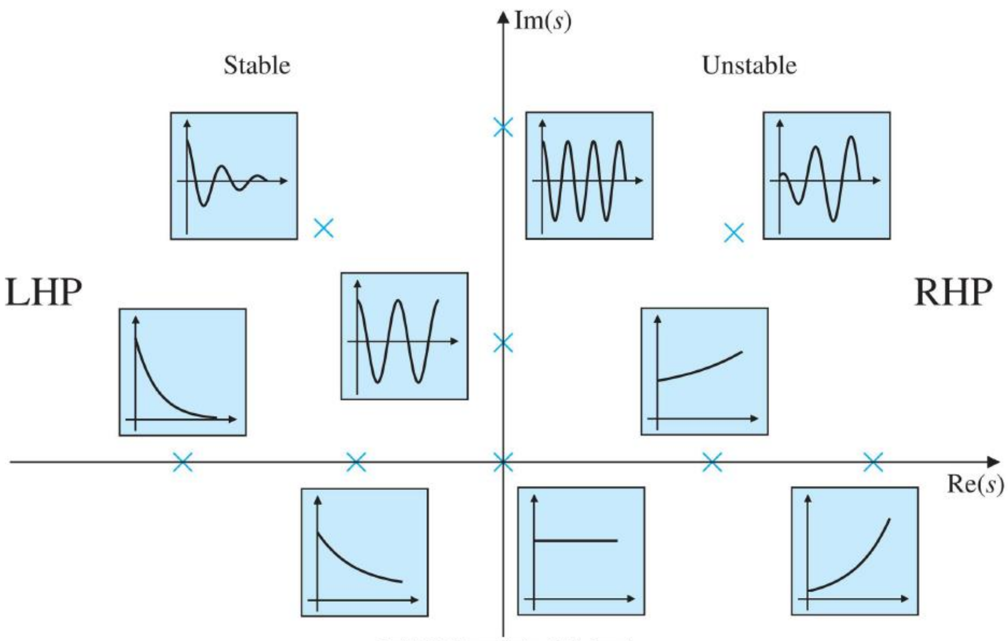

Performance Equations for the Step Response of an Underdamped Second-Order System

Performance Criteria	Equation
Peak time, t_p	$t_p = \dfrac{\pi}{\omega_n\sqrt{1-\zeta^2}}$
Maximum overshoot, M_{os}	$M_{os} = e^{-\zeta\pi/\sqrt{1-\zeta^2}}$
Settling time to steady state, t_s	$t_s = \dfrac{4}{\zeta\omega_n}$
Period of oscillation, T_{period}	$T_{period} = \dfrac{2\pi}{\omega_n\sqrt{1-\zeta^2}}$
Number of cycles to steady state, N_{cycles}	$N_{cycles} = \dfrac{2\sqrt{1-\zeta^2}}{\pi\zeta}$

Steady-State Errors for Closed-Loop Control Systems with Unity Feedback

System Type, N	Unit-Step Input $r(t) = 1$	Unit-Ramp Input $r(t) = t$	Unit-Parabola Input $r(t) = t^2/2$
0	$\dfrac{1}{1+K_{sp}}$	∞	∞
1	0	$\dfrac{1}{K_{sv}}$	∞
2	0	0	$\dfrac{1}{K_{sa}}$

Static position error constant: $K_{sp} = \lim_{s\to 0} G(s)$.
Static velocity error constant: $K_{sv} = \lim_{s\to 0} sG(s)$.
Static acceleration error constant: $K_{sa} = \lim_{s\to 0} s^2G(s)$.

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Consider the system shown in Figure 5–6, where $\zeta = 0.6$ and $\omega_n = 5$ rad/sec. Let us obtain the rise time t_r, peak time t_p, maximum overshoot M_p, and settling time t_s when the system is subjected to a unit-step input.

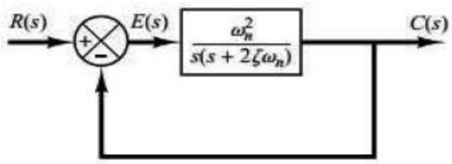

Figure 5–6
Second-order system.

Sol.

From the given values of ζ and ω_n, we obtain $\omega_d = \omega_n\sqrt{1-\zeta^2} = 4$ and $\sigma = \zeta\omega_n = 3$.

Rise time t_r: The rise time is

$$t_r = \frac{\pi - \beta}{\omega_d} = \frac{3.14 - \beta}{4}$$

where β is given by

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1}\frac{4}{3} = 0.93 \text{ rad}$$

The rise time t_r is thus

$$t_r = \frac{3.14 - 0.93}{4} = 0.55 \text{ sec}$$

Peak time t_p: The peak time is

$$t_p = \frac{\pi}{\omega_d} = \frac{3.14}{4} = 0.785 \text{ sec}$$

Maximum overshoot M_p: The maximum overshoot is

$$M_p = e^{-(\sigma/\omega_d)\pi} = e^{-(3/4)\times 3.14} = 0.095$$

The maximum percent overshoot is thus 9.5%.

Settling time t_s: For the 2% criterion, the settling time is

$$t_s = \frac{4}{\sigma} = \frac{4}{3} = 1.33 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = \frac{3}{3} = 1 \text{ sec}$$

1. For the system shown in Figure 5–13(a), determine the values of gain K and velocity-feedback constant Kh so that the maximum overshoot in the unit-step response is 0.2 and the peak time is 1 sec. With these values of K and Kh, obtain the rise time and settling time. Assume that J=1 kg-m2 and B=1 N-m/rad/sec. Determination of the values of K and Kh: The maximum overshoot Mp is given by Equation

$$M_p = e^{-(\zeta/\sqrt{1-\zeta^2})\pi}$$

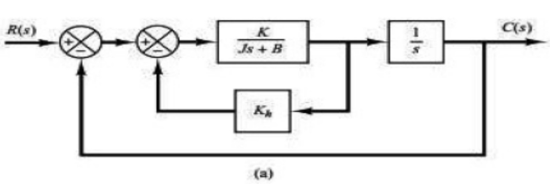

FIG. 5-13

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The maximum overshoot is given,

This value must be 0.2. Thus,

$$e^{-(\zeta/\sqrt{1-\zeta^2})\pi} = 0.2$$

or

$$\frac{\zeta\pi}{\sqrt{1-\zeta^2}} = 1.61$$

which yields

$$\zeta = 0.456$$

The peak time t_p is specified as 1 sec; therefore, from Equation (5–20),

$$t_p = \frac{\pi}{\omega_d} = 1$$

or

$$\omega_d = 3.14$$

Since ζ is 0.456, ω_n is

$$\omega_n = \frac{\omega_d}{\sqrt{1-\zeta^2}} = 3.53$$

Since the natural frequency ω_n is equal to $\sqrt{K/J}$,

$$K = J\omega_n^2 = \omega_n^2 = 12.5 \text{ N-m}$$

Then K_h is, from Equation (5–25),

$$K_h = \frac{2\sqrt{KJ}\zeta - B}{K} = \frac{2\sqrt{K}\zeta - 1}{K} = 0.178 \text{ sec}$$

Rise time t_r: From Equation (5–19), the rise time t_r is

$$t_r = \frac{\pi - \beta}{\omega_d}$$

where

$$\beta = \tan^{-1}\frac{\omega_d}{\sigma} = \tan^{-1} 1.95 = 1.10$$

Thus, t_r is

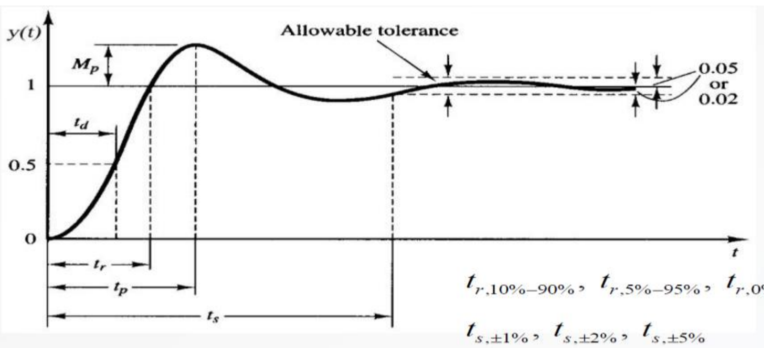
$$t_r = 0.65 \text{ sec}$$

Settling time t_s: For the 2% criterion,

$$t_s = \frac{4}{\sigma} = 2.48 \text{ sec}$$

For the 5% criterion,

$$t_s = \frac{3}{\sigma} = 1.86 \text{ sec}$$



$t_{r,10\%-90\%}$, $t_{r,5\%-95\%}$, $t_{r,0\%-100\%}$

$t_{s,\pm1\%}$, $t_{s,\pm2\%}$, $t_{s,\pm5\%}$

$$\%M_p = \frac{y(t_p) - y(\infty)}{y(\infty)} \cdot 100\%$$

Example:

Find the correlation between the poles and the impulse response of the following system, and further find the exact impulse response.

$$H(s)=\frac{2s+1}{s^2+2s+5}$$

Since $H(s)=\dfrac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}$, $\omega_n^2=5 \Rightarrow \omega_n=\sqrt{5}=2.24 \text{ rad/sec}$

$$2\zeta\omega_n=2 \Rightarrow \zeta=0.447$$

The exact response can be otained from:

$$H(s)=\frac{2s+1}{s^2+2s+5}=\frac{2s+1}{(s+1)^2+2^2} \quad \Rightarrow \text{poles at } s=-1\pm j2$$

To find the inverse Laplace transform, the righthand side of the last equation is broken into two parts:

$$H(s)=\frac{2s+1}{(s+1)^2+2^2}$$

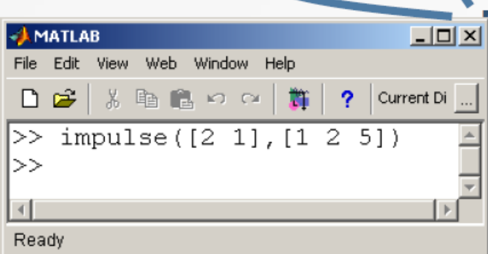
$$=2\frac{s+1}{(s+1)^2+2^2}-\frac{1}{2}\frac{2}{(s+1)^2+2^2}$$

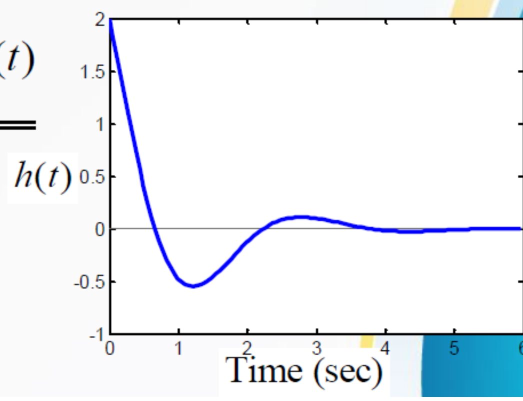
$$h(t)=\mathcal{L}^{-1}\left(H(s)\right)$$

$$=\left(2e^{-t}\cos 2t-\frac{1}{2}e^{-t}\sin 2t\right)\cdot 1(t)$$

Damped sinusoidal oscillation

$f(t)$	$F(s)$
$\sin\omega t\ 1(t)$	$\frac{\omega}{s^2+\omega^2}$
$\cos\omega t\ 1(t)$	$\frac{s}{s^2+\omega^2}$
$e^{-at}\sin\omega t\ 1(t)$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos\omega t\ 1(t)$	$\frac{s+a}{(s+a)^2+\omega^2}$





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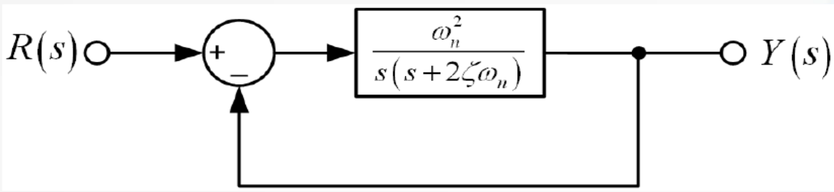
Example:

Consider a system shown below with $\zeta = 0.6$ and $\omega_n = 5$ rad/s. Obtain the rise time, peak time, maximum overshoot, and settling time of the system when it is subjected to a unit step input.



After block diagram simplification,

$$\frac{Y(s)}{R(s)} = \frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

Standard form of second-order system

$$\zeta = 0.6, \omega_n = 5 \text{ rad/s} \Rightarrow \omega_d = \sqrt{1-\zeta^2}\,\omega_n = \sqrt{1-0.6^2}\cdot 5 = 4 \text{ rad/s}$$

$$\Rightarrow \sigma = \zeta\omega_n = 0.6\cdot 5 = 3 \text{ rad/s}$$

$$t_r = \frac{1}{\omega_d}\tan^{-1}\left(-\frac{\omega_d}{\sigma}\right)$$ **In second quadrant**

$$= \frac{1}{4}\tan^{-1}\left(-\frac{4}{3}\right) = \frac{1}{4}(\pi - 0.927) = \underline{\underline{0.554 \text{ s}}}$$

$$t_p = \frac{\pi}{\omega_d} = \frac{\pi}{4} = \underline{\underline{0.785 \text{ s}}}$$

$$M_p = y(t_p) - y(\infty) = (1 + e^{-\zeta\pi/\sqrt{1-\zeta^2}}) - 1$$

$$M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}} = e^{-(0.6\cdot\pi)/0.8} = \underline{\underline{0.0948}}$$

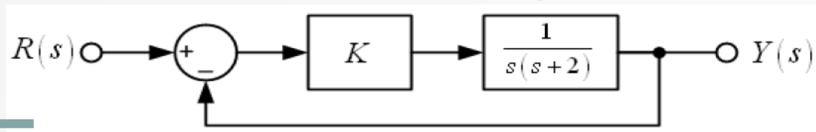
$$\%M_p = e^{-\zeta\pi/\sqrt{1-\zeta^2}}\cdot 100\% = \underline{\underline{9.48\%}}$$

$$t_{s,\pm 2\%} = \frac{4}{\zeta\omega_n} = \frac{4}{0.6\cdot 5} = \underline{\underline{1.333 \text{ s}}}$$

$$t_{s,\pm 5\%} = \frac{3}{\zeta\omega_n} = \frac{3}{0.6\cdot 5} = \underline{\underline{1 \text{ s}}}$$

Example:

For the unity feedback system shown below, specify the gain K of the proportional controller so that the output $y(t)$ has an overshoot of no more than 10% in response to a unit step.



$$\frac{Y(s)}{R(s)} = \frac{\frac{K}{s(s+2)}}{1+\frac{K}{s(s+2)}} = \frac{K}{s^2+2s+K} \quad \Rightarrow 2\zeta\omega_n = 2$$

$$\Rightarrow \omega_n^2 = K$$

$$\%M_p \le 10\% \Rightarrow e^{-\zeta\pi/\sqrt{1-\zeta^2}} \le 0.1 \quad \Rightarrow \zeta \ge 0.592$$

$$\Rightarrow \omega_n = \frac{1}{\zeta} \le \frac{1}{0.592} = 1.689$$

$$\Rightarrow K = \omega_n^2 \le 1.689^2 = 2.853$$

$$\therefore \underline{\underline{0 < K \le 2.853}}$$