Roots of Algebraic and transcendental Equations

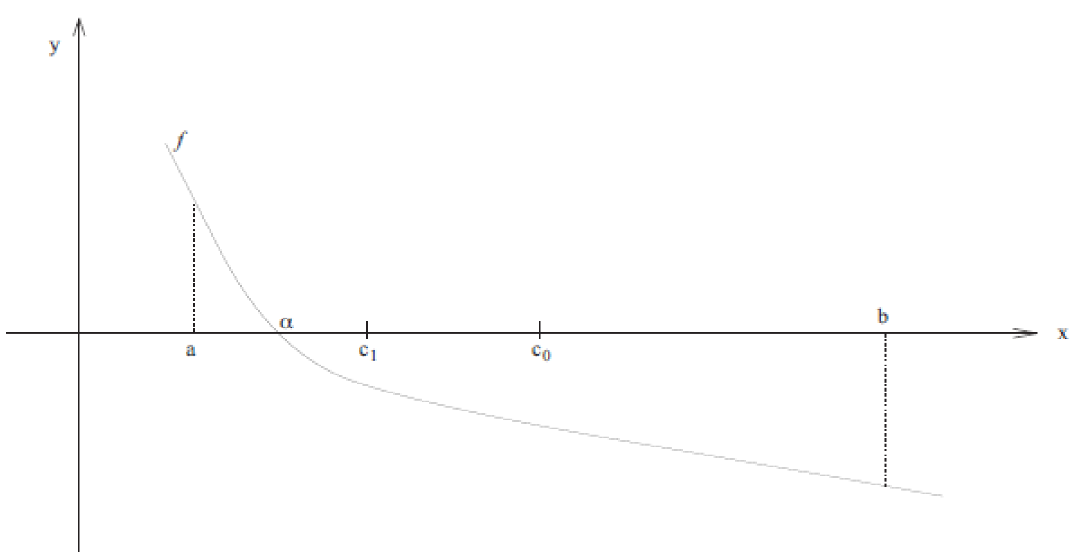
In this Lecture ,we shall discuss one of the oldest approximation problems which consists of finding the roots of an equation. It is also one of the most commonly occurring problems in applied mathematics. The root-finding problem consists of the following: given a continuous function f, find the values of x that satisfy the equation $f(x) = 0$.

The solutions of this equation are called the zeros of f or the roots of the equation.

The Bisection Method

Let $f(x)$ *be a given function, continuous on an interval* $[a, b]$*, such that* $f(a).f(b) < 0$.

Using the Inter mediate Value Theorem, it follows that there exists at least one zero of f in (a, b)*. To simplify your discussion, we assume that f has exactly one root* α*. Such a function is shown in Figure 1. The bisection method is based on halving the interval* $[a, b]$ *to determine a smaller and smaller interval with in which* α *must lie.*



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The procedure is carried out

- *Defining the midpoint of [a, b], c = (a + b)/2 and then computing the product f(c)*f(b).*
- *If the product is negative, then the root is in the interval [c, b].*
- *If the product is positive, then the root is in the interval [a, c]. The process of halving the new interval continues until the root is located as accurately as desired, that is* $|an - bn| < \epsilon$

***Example 1**/ The function f(x) =* $x^3 - x^2 - 1$ *has exactly one zero in [1, 2]. Use the bisection algorithm to approximate the zero of f to within* 10^{-4}.

Since $f(1) = -1 < 0$ and $f(2) = 3 > 0$, then (3.4) is satisfied. Starting with $a_0 = 1$ and $b_0 = 2$, we compute

$$c_0 = \frac{a_0 + b_0}{2} = \frac{1+2}{2} = 1.5 \quad \text{and} \quad f(c_0) = 0.125.$$

Since $f(1.5)f(2) > 0$, the function changes sign on $[a_0, c_0] = [1, 1.5]$.
To continue, we set $a_1 = a_0$ and $b_1 = c_0$; so

$$c_1 = \frac{a_1 + b_1}{2} = \frac{1+1.5}{2} = 1.25 \quad \text{and} \quad f(c_1) = -0.609375.$$

Again $f(1.25)f(1.5) < 0$ so the function changes sign on $[c_1, b_1] = [1.25, 1.5]$. Next we set $a_2 = c_1$ and $b_2 = b_1$. Continuing in this manner leads to the values in Table 1,

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iter	a	b	c	f(c)	lb-al/2
0	1.0000	2.0000	1.500000	0.125000	0.500000
1	1.0000	1.5000	1.250000	-0.609375	0.250000
2	1.2500	1.5000	1.375000	-0.291016	0.125000
3	1.3750	1.5000	1.437500	-0.095947	0.062500
4	1.4375	1.5000	1.468750	0.011200	0.031250
5	1.4375	1.4688	1.453125	-0.043194	0.015625
6	1.4531	1.4688	1.460938	-0.016203	0.007813
7	1.4609	1.4688	1.464844	-0.002554	0.003906
8	1.4648	1.4688	1.466797	0.004310	0.001953
9	1.4648	1.4668	1.465820	0.000875	0.000977
10	1.4648	1.4658	1.465332	-0.000840	0.000488
11	1.4653	1.4658	1.465576	0.000017	0.000244
12	1.4653	1.4656	1.465454	-0.000411	0.000122

The False Position Method

*This is the oldest method for finding the real root of a nonlinear equation f(x)= and closely resembles the bisection method. In this method, also known as regula falsi or the method of chords, we choose two points **a** and **b** such that f(a) and f(b) are of opposite signs. Hence, a root must lie in between these points. Now, the equation of the chord joining the two points [a,f(a)] and [b,f(b)] is given by:*

$$y - f(a) = \frac{f(b)-f(a)}{b-a}\,(x-a) \qquad (1)$$

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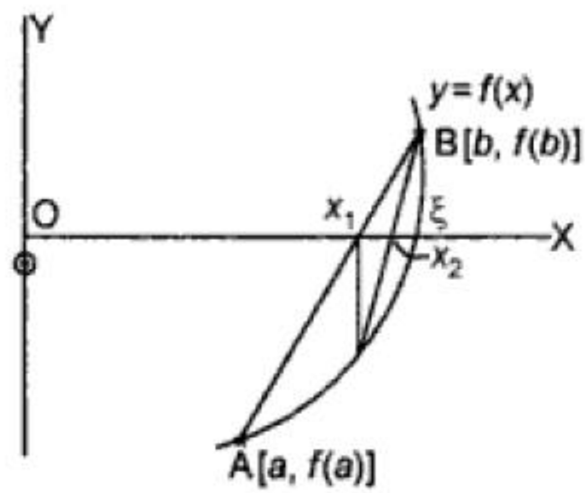
The method consists in replacing the part of the curve between the points [a,f(a)] and [b,f(b)]by means of the chord joining these points, and taking the point of intersection of the chord with the x-axis as an approximation to the root. The point of intersection in the present case is obtained by putting y=0 in Eq(1). Thus, we obtain

$$x_1 = a - \frac{f(a)}{f(b)-f(a)}(b-a) = \frac{af(b)-bf(a)}{f(b)-f(a)} \qquad (2)$$

*which is the first approximation to the root of f(x)=0. If now f(x_1) and f(a) are of opposite signs, then the root lies between **a** and **x_1**, and we replace **b** by **x_1** in (2), and obtain the next approximation. Otherwise, we replace **a** by **x_1** and generate the next approximation. The procedure is repeated until the root is obtained to the desired accuracy.*



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Example 2. Find a real root of the equation :

$$f(x) = x^3 - 2x - 5 = 0.$$

We find $f(2) = -1$ and $f(3) = 16$. Hence $a = 2$, $b = 3$, and a root lies between 2 and 3. Equation (2.7) gives

$$x_1 = \frac{2(16) - 3(-1)}{16 - (-1)} = \frac{35}{17} = 2.058823529.$$

Now, $f(x_1) = -0.390799917$ and hence the root lies between 2.058823529 and 3.0. Using formula (2.7), we obtain

$$x_2 = \frac{2.058823529(16) - 3(-0.390799917)}{16.390799917} = 2.08126366.$$

Since $f(x_2) = -0.147204057$, it follows that the root lies between 2.08126366 and 3.0. Hence, we have

$$x_3 = \frac{2.08126366(16) - 3(-0.147204057)}{16.147204057} = 2.089639211.$$

Proceeding in this way, we obtain successively:

$$x_4 = 2.092739575, \qquad x_5 = 2.09388371,$$

$$x_6 = 2.094305452, \qquad x_7 = 2.094460846, \ldots$$

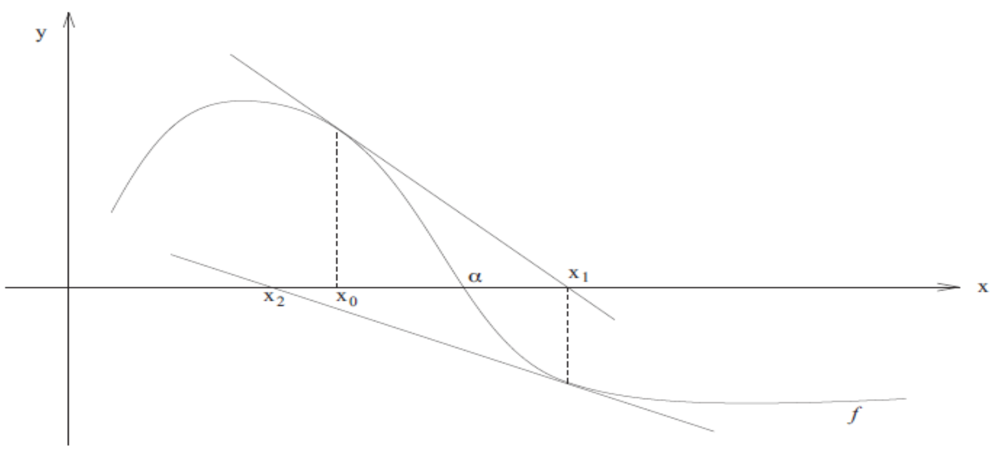
The Newton's Method

This method is generally used to improve the result obtained by one of the previous methods. To use the method, we begin with an initial guess x_0, sufficiently close to the root α. The point at which the tangent line to f at f (x_0, f (x_0)) crosses the x-axis gives the next approximation x_1. It is clear that the value x_1 is much closer to α than the original guess x_0.

If x_{n+1} denotes the value obtained by the succeeding iterations, that is the x-intercept of the tangent line to f at (x_n , $f(x_n)$), then a formula relating x_n and x_{n+1}, known as Newton's method, is given by

$$x_{n+1} = x_n - \frac{f(x_n)}{\grave{f}(x_n)}$$

provided $\grave{f}(x_n)$ is not zero.



Example 1 Use the Newton–Raphson method to find a root of the equation $x^3 - 2x - 5 = 0$.

Here $f(x) = x^3 - 2x - 5$ and $f'(x) = 3x^2 - 2$. Hence Eq. (2.24) gives:

$$x_{n+1} = x_n - \frac{x_n^3 - 2x_n - 5}{3x_n^2 - 2} \qquad \text{(i)}$$

Choosing $x_0 = 2$, we obtain $f(x_0) = -1$ and $f'(x_0) = 10$. Putting $n = 0$ in (i), we obtain

$$x_1 = 2 - \left(-\frac{1}{10}\right) = 2.1$$

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Now,

$$f(x_1) = (2.1)^3 - 2(2.1) - 5 = 0.061,$$

and

$$f'(x_1) = 3(2.1)^2 - 2 = 11.23.$$

Hence

$$x_2 = 2.1 - \frac{0.061}{11.23} = 2.094568.$$

Example 2. Find a root of the equation $x \sin x + \cos x = 0$.

We have

$$f(x) = x \sin x + \cos x \quad \text{and} \quad f'(x) = x \cos x.$$

The iteration formula is therefore

$$x_{n+1} = x_n - \frac{x_n \sin x_n + \cos x_n}{x_n \cos x_n}.$$

With $x_0 = \pi$, the successive iterates are given below

n	x_n	$f(x_n)$	x_{n+1}
0	3.1416	–1.0	2.8233
1	2.8233	–0.0662	2.7986
2	2.7986	–0.0006	2.7984
3	2.7984	0.0	2.7984

EXERCISES

Obtain a root, correct to three decimal places, for each of the following equations using the bisection method (Problems 1–10):

2.1. $x^3 + x^2 + x + 7 = 0$

2.2. $x^3 - 4x - 9 = 0$

2.3. $x^3 - x - 4 = 0$

2.4. $x^3 - 18 = 0$

2.5. $x^3 - x^2 - 1 = 0$

2.6. $x^3 + x^2 - 1 = 0$

2.7. $x^3 - 3x - 5 = 0$

2.8. $x^3 - x - 1 = 0$

2.9. $x^3 - 5x + 3 = 0$

2.10. $x^3 + x - 1 = 0.$

Use the method of false position to obtain a root, correct to three decimal places, of each of the following equations (Problems 11–15):

2.11. $x^3 + x^2 + x + 7 = 0$

2.12. $x^3 - x - 4 = 0$

2.13. $x^3 - x^2 - 1 = 0$

2.14. $x^3 - x - 1 = 0$

2.15. $x^3 + x - 1 = 0.$

Use Newton–Raphson method to obtain a root, correct to three decimal places, of the following equations (Problems 27–36):

2.27. $x^{\sin 2} - 4 = 0$

2.28. $\sin x = 1 - x$

2.29. $x^3 - 5x + 3 = 0$

2.30. $x^4 + x^2 - 80 = 0$

2.31. $x^3 + 3x^2 - 3 = 0$

2.32. $4(x - \sin x) = 1$

2.33. $x - \cos x = 0$

2.34. $\sin x = (1/2)\, x$

2.35. $x + \log x = 2$

2.36. $xe^{-2x} = (1/2) \sin x$

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