Indeterminate Forms and L'Hôpital's Rule

John Bernoulli discovered a rule for calculating limits of fractions whose numerators and denominators both approach zero or ∞. The rule is known today as **L'Hôpital's Rule**

THEOREM L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

EXAMPLE 1 Using L'Hôpital's Rule

(a) $\lim_{x \to 0} \frac{3x - \sin x}{x} = \left. \frac{3 - \cos x}{1} \right|_{x=0} = 2$

(b) $\lim_{x \to 0} \frac{\sqrt{1+x} - 1}{x} = \left. \frac{\frac{1}{2\sqrt{1+x}}}{1} \right|_{x=0} = \frac{1}{2}$

EX-1 – **Evaluate the following limits :**

1) $\lim_{x \to o} \frac{\sin x}{x}$ 2) $\lim_{x \to 2} \frac{\sqrt{x^2 + 5} - 3}{x^2 - 4}$

3) $\lim_{x \to 0} \frac{x - \sin x}{x^3}$ 4) $\lim_{x \to \frac{\pi}{2}} -(x - \frac{\pi}{2}).\tan x$

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Sol. –

1) $$\lim_{x\to 0}\frac{\sin x}{x} \Rightarrow \frac{0}{0} \quad using\ L'Hoptal's\ rule \Rightarrow$$
$$= \lim_{x\to 0}\frac{\cos x}{1} = \cos 0 = 1$$

2) $$\lim_{x\to 2}\frac{\sqrt{x^2+5}-3}{x^2-4} \Rightarrow \frac{0}{0} \quad using\ L'Hoptal's\ rule \Rightarrow$$
$$= \lim_{x\to 2}\frac{\frac{x}{\sqrt{x^2+5}}}{2x} = \lim_{x\to 2}\frac{1}{2\sqrt{x^2+5}} = \frac{1}{2\sqrt{4+5}} = \frac{1}{6}$$

3) $$\lim_{x\to 0}\frac{x-\sin x}{x^3} \Rightarrow \frac{0}{0} \quad using\ L'Hoptal's\ rule \Rightarrow$$
$$= \lim_{x\to 0}\frac{1-\cos x}{3x^2} \Rightarrow \frac{0}{0} \quad using\ L'Hopital's\ rule \Rightarrow$$
$$= \frac{1}{6}\lim_{x\to 0}\frac{\sin x}{x} = \frac{1}{6}$$

4) $$\lim_{x\to\frac{\pi}{2}} -(x-\frac{\pi}{2})\tan x \Rightarrow 0.\infty \quad we\ can't\ using\ L'Hoptal's\ rule \Rightarrow$$
$$= \lim_{x\to\frac{\pi}{2}} -\frac{x-\frac{\pi}{2}}{\cos x} . \lim_{x\to\frac{\pi}{2}} \sin x \Rightarrow \frac{0}{0} \quad using\ L'Hopital's\ rule \Rightarrow$$
$$= \lim_{x\to\frac{\pi}{2}} -\frac{1}{-\sin x} . \lim_{x\to\frac{\pi}{2}} \sin x = \frac{1}{\sin\frac{\pi}{2}} . \sin\frac{\pi}{2} = 1$$

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The slope of the curve

Secant to the curve is a line through two points on a curve. Slopes and tangent lines:

1. we start with what we can calculate , namely the slope of secant through P and a point Q nearby on the curve .

2. we find the limiting value of the secant slope (if it exists) as Q approaches p along the curve .

3. we take this number to be the slope of the curve at P and define the tangent to the curve at P to be the line through p with this slope .

The derivative of the function f is the slope of the curve :

$$\text{the slope} = m = f'(x) = \frac{dy}{dx}$$

EX-2- **Write an equation for the tangent line at $x = 3$ of the curve :**

$$f(x) = \frac{1}{\sqrt{2x+3}}$$

Sol.-

$$m = f'(x) = -\frac{1}{\sqrt{(2x+3)^3}} \Rightarrow [m]_{x=3} = f'(3) = -\frac{1}{27}$$

$$f(3) = \frac{1}{\sqrt{2*3+3}} = \frac{1}{3}$$

The equation of the tangent line is :

$$y - \frac{1}{3} = -\frac{1}{27}(x-3) \Rightarrow 27y + x = 12$$

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Velocity and acceleration and other rates of changes

The average velocity of a body moving along a line is :

$$v_{av} = \frac{\Delta s}{\Delta t} = \frac{f(t + \Delta t) - f(t)}{\Delta t}$$

The instantaneous velocity of a body moving along a line is the derivative of its position $s = f(t)$ *with respect to time t .*

i.e

$$v = \frac{ds}{dt} = \lim_{\Delta x \to 0} \frac{\Delta s}{\Delta t}$$

The rate at which the particle's velocity increase is called its acceleration ***a*** *. If a particle has an initial velocity* **v** *and a constant acceleration a, then its velocity after time* ***t*** *is* ***v + at*** .

$$average\ acceleration = a_{av} = \frac{\Delta v}{\Delta t}$$

The acceleration at an instant is the limit of the average acceleration for an interval following that instant , as the interval tends to zero .

i.e.

$$a = \frac{dv}{dt} = \lim_{\Delta x \to 0} \frac{\Delta v}{\Delta t}$$

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The average rate of a change in a function y = f (x) over the interval from x to $x + \Delta x$

$$\boldsymbol{average\ rate\ of\ change} = \frac{\boldsymbol{f(x+\Delta x) - f(x)}}{\boldsymbol{\Delta x}}$$

The instantaneous rate of change of $\boldsymbol{f}$ *at* **x** *is the derivative.*

$$\boldsymbol{f' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}}$$

EX-1- *The position s (in meters) of a moving body as a function of time t (secondis) is :* $s = 2t^5 + 5t - 3$ *; find :*

a) The displacement and average velocity for the time interval from t = 0 to t = 2 seconds .

b) The body's velocity at t = 2 seconds .

Sol.-

a) 1) $\Delta s = s(t+\Delta t) - s(t) = 2(t+\Delta t)^2 + 5(t+\Delta t) - 3 - [2t^2 + 5t - 3]$

$= (4t+5)\Delta t + 2(\Delta t)^2$

at $t = 0$ *and* $\Delta t = 2 \Rightarrow \Delta s = (4*0+5)*2 + 2*2^2 = 18$

2) $v_{av} = \frac{\Delta s}{\Delta t} = \frac{(4t+5)\Delta t + 2(\Delta t)^2}{\Delta t} = 4t + 5 + 2.\Delta t$

at $t = 0$ *and* $\Delta t = 2 \Rightarrow v_{av} = 4*0 + 5 + 2*2 = 9$

b) $v(t) = \frac{d}{dt} f(t) = 4t + 5$

$v(2) = 4*2 + 5 = 13$

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EX-4- *A particle moves along a straight line so that after t (seconds) , its distance from O a fixed point on the line is s (meters) , where:* $s = t^3 - 3t^2 + 2t$

i) when is the particle at O ?

ii) what is its velocity and acceleration at these times ?

iii) what is its average velocity during the first second ?

iv) what is its average acceleration between t = 0 and t = 2 ?

Sol. –

$$i)\quad at\ \ s = 0 \Rightarrow t^3 - 3t^2 + 2t = 0 \Rightarrow t(t-1)(t-2) = 0$$

$$either\ \ t = 0\ \ or\ \ t = 1\ \ or\ \ t = 2\ \ sec.$$

$$ii)\quad velocity = v(t) = 3t^2 - 6t + 2 \Rightarrow v(0) = 2m/s$$

$$\Rightarrow v(1) = -1m/s$$

$$\Rightarrow v(2) = 2m/s$$

$$acceleration = a(t) = 6t - 6 \Rightarrow a(0) = -6m/s^2$$

$$\Rightarrow a(1) = 0m/s^2$$

$$\Rightarrow a(2) = 6m/s^2$$

$$iii)\quad v_{av} = \frac{\Delta s}{\Delta t} = \frac{s(1) - s(0)}{1 - 0} = \frac{1 - 3 + 2 - 0}{1} = 0m/s$$

$$iv)\quad a_{av} = \frac{\Delta v}{\Delta t} = \frac{v(2) - v(0)}{2 - 0} = \frac{2 - 2}{2} = 0m/s^2$$

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Exercises

1-Find the limits for the following functions by using L'Hopital's rule :

$$1)\ \lim_{x\to\infty}\frac{5x^2-3x}{7x^2+1} \qquad 2)\ \lim_{t\to 0}\frac{\sin t^2}{t}$$

$$3)\ \lim_{x\to\frac{\pi}{2}}\frac{2x-\pi}{\cos x} \qquad 4)\ \lim_{t\to 0}\frac{\cos t-1}{t^2}$$

$$5)\ \lim_{x\to\frac{\pi}{2}}\frac{1-\sin x}{1+\cos 2x} \qquad 6)\ \lim_{x\to\frac{\pi}{4}}\frac{\sin x-\cos x}{x-\frac{\pi}{4}}$$

$$7)\ \lim_{x\to 1}\frac{2x^2-(3x+1)\sqrt{x}+2}{x-1} \qquad 8)\ \lim_{x\to 0}\frac{x(\cos x-1)}{\sin x-x}$$

$$9)\ \lim_{x\to 0} x.\csc^2\sqrt{2x} \qquad 10)\ \lim_{x\to 0}\frac{\sin x^2}{x.\sin x}$$

$$(ans.:1)\frac{5}{7};2)0;3)-2;4)-\frac{1}{2};5)\frac{1}{4};6)\sqrt{2};7)-1;8)3;9)\frac{1}{2};10)1)$$

2- Find the velocity *v* if a particle's position at time *t* is $s=180t-16t^2$.When does the velocity vanish ? *(ans.: 5.625)*

3-If a ball is thrown straight up with a velocity of *32* ft./sec. , its high after *t* sec. is given by the equation $s=32t-16t^2$. At what instant will the ball be at its highest point ? and how high will it rise ? *(ans.: 1, 16)*

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3- A stone is thrown vertically upwards at *35* m./sec. . Its height is : $s = 35t - 4.9t^2$ in meter above the point of projection where *t* is time in second later :

a) What is the distance moved, and the average velocity during the *3rd* sec. (from $t = 2$ to $t = 3$) ?

b) Find the average velocity for the intervals ($t = 2$ to $t = 2.5$) , ($t = 2$ to $t = 2.1$) .

(ans.: a) 10.5 , 10.5 ; b) 12.95, 14.91

4- A stone is thrown vertically upwards at *24.5* m./sec. from a point on the level with but just beyond a cliff ledge . Its height above the ledge *t* sec. later is $4.9t\,(5 - t)$ m. . If its velocity is *v* m./sec. , differentiate to find *v* in terms of *t* :

i) when is the stone at the ledge level ?

ii) find its height and velocity after *1* , *2* , *3* , and *6* sec.

iii) what meaning is attached to negative value of *s* ? a negative value of *v* ?

iv) when is the stone momentarily at rest ? what is the greatest height reached ?

v) find the total distance moved during the *3rd* sec. .

(ans.:v=24.5-9.8t; i)0,5; ii)19.6,29.4,29.4,-29.4;14.7,4.9, -4.9,-34.3; iv)2.5;30.625; v)2.45)

5-A particle moves in a straight line so that after t sec. it is *s* m. , from a fixed point *O* on the line , where $s = t^4 + 3t^2$. Find :

i) The acceleration when $t = 1$, $t = 2$, and $t = 3$.

ii) The average acceleration between $t = 1$ and $t = 3$.

(ans.: i)18, 54,114; ii)58)

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