GAUSS ELEMINATION METHOD

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Engineers often need to solve large systems of linear equations; for example in determining the forces in a large framework or finding currents in a complicated electrical circuit. The method of Gauss elimination provides a systematic approach to their solution.

EXAMPLE 1:
Solve The Following Three Equation In Three Unknowns:

The easiest set of three simultaneous linear equations to solve is of the following type:

$$3x_1 = 6,$$

$$2x_2 = 5,$$

$$4x_3 = 7$$

Now we must convert the simillanouse equation to matrix form as follows:

$$\begin{bmatrix} 3 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 6 \\ 5 \\ 7 \end{bmatrix}$$

That is mean $x_1 = \frac{6}{3} = 2$ and $x_2 = \frac{5}{2} = 2.5$ and $x_3 = \frac{7}{4}$

Example 2:

Solve the following equations:

$$\begin{aligned} 3x_1 + x_2 - x_3 &= 0 \\ 2x_2 + x_3 &= 12 \\ 3x_3 &= 6. \end{aligned}$$

The last equation can be solved immediately to give $x_3 = 2$.
Substituting this value of x_3 into the second equation gives

$$2x_2 + 2 = 12 \qquad \text{from which} \qquad 2x_2 = 10 \qquad \text{so that} \qquad x_2 = 5$$

Substituting these values of x_2 and x_3 into the first equation gives

$$3x_1 + 5 - 2 = 0 \qquad \text{from which} \qquad 3x_1 = -3 \qquad \text{so that} \qquad x_1 = -1$$

Hence the solution is $[x_1, x_2, x_3]^T = [-1, 5, 2]^T$.
This process of solution is called **back-substitution**.
In matrix form the system of equations is

$$\begin{bmatrix} 3 & 1 & -1 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 12 \\ 6 \end{bmatrix}.$$

The matrix of coefficients is said to be **upper triangular** because all elements below the leading diagonal are zero. Any system of equations in which the coefficient matrix is triangular (whether upper or lower) will be particularly easy to solve.

Example 3:

Solve the following equations by using back substitution:

$$\begin{bmatrix} 2 & -1 & 3 \\ 0 & 3 & -1 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 7 \\ 5 \\ 2 \end{bmatrix}.$$

OR :

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= 7 \\ 3x_2 - x_3 &= 5 \\ 2x_3 &= 2 \end{aligned}$$

$x_2 = 2, \quad x_1 = 3$. Therefore the solution is $x_1 = 3, x_2 = 2$ and $x_3 = 1$.

The general system of three simultaneous linear equations

In the previous subsection we met systems of equations which could be solved by back-substitution alone. In this Section we meet systems which are not so amenable and where preliminary work must be done before back-substitution can be used. Consider the system

$$
\begin{aligned}
x_1 + 3x_2 + 5x_3 &= 14 \\
2x_1 - x_2 - 3x_3 &= 3 \\
4x_1 + 5x_2 - x_3 &= 7
\end{aligned}
$$

We will use the solution method known as **Gauss elimination**, which has three stages. In the first stage the equations are written in matrix form. In the second stage the matrix equations are replaced by a system of equations having the same solution but which are in **triangular form**. In the final stage the new system is solved by **back-substitution**.

Stage 1: Matrix Formulation

The first step is to write the equations in matrix form:

$$
\begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & -3 \\ 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 14 \\ 3 \\ 7 \end{bmatrix}.
$$

Then, for conciseness, we combine the matrix of coefficients with the column vector of right-hand sides to produce the **augmented matrix**:

$$
\left[\begin{array}{ccc|c} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right]
$$

If the general system of equations is written $AX = B$ then the augmented matrix is written $[A|B]$.

Hence the first equation

$$x_1 + 3x_2 + 5x_3 = 14$$

is replaced by the first row of the augmented matrix,

$$1 \quad 3 \quad 5 \quad | \quad 14$$ and so on.

Stage 1 has now been completed. We will next triangularise the matrix of coefficients by means of **row operations**. There are three possible row operations:

- interchange two rows;
- multiply or divide a row by a non-zero constant factor;
- add to, or subtract from, one row a multiple of another row.

Note that interchanging two rows of the augmented matrix is equivalent to interchanging the two corresponding equations. The shorthand notation we use is introduced by example. To interchange row 1 and row 3 we write $R1 \leftrightarrow R3$. To divide row 2 by 5 we write $R2 \div 5$. To add three times row 1 to row 2, we write $R2 + 3R1$. In the Task which follows you will see where these annotations are placed.

Note that these operations neither create nor destroy solutions so that at every step the system of equations has the same solution as the original system.

Stage 2: Triangularisation

The second stage proceeds by first eliminating x_1 from the second and third equations using row operations.

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 2 & -1 & -3 & 3 \\ 4 & 5 & -1 & 7 \end{array}\right] \begin{array}{l} \\ R2 - 2\times R1 \\ R3 - 4\times R1 \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right]$$

In the above we have subtracted twice row (equation) 1 from row (equation) 2.
In full these operations would be written, respectively, as

$$(2x_1 - x_2 - 3x_3) - 2(x_1 + 3x_2 + 5x_3) = 3 - 2\times 14 \quad \text{or} \quad -7x_2 - 13x_3 = -25$$

and

$$(4x_1 + 5x_2 - x_3) - 4(x_1 + 3x_2 + 5x_3) = 7 - 4\times 14 \quad \text{or} \quad -7x_2 - 21x_3 = -49.$$

Now since all the elements in rows 2 and 3 are negative we multiply throughout by -1:

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & -7 & -13 & -25 \\ 0 & -7 & -21 & -49 \end{array}\right] \begin{array}{l} \\ R2\times(-1) \\ R3\times(-1) \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 7 & 21 & 49 \end{array}\right]$$

Finally, we eliminate x_2 from the third equation by subtracting equation 2 from equation 3 i.e. $R3 - R2$:

$$\left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 7 & 21 & 49 \end{array}\right] \begin{array}{l} \\ \\ R3 - R2 \end{array} \quad \Rightarrow \quad \left[\begin{array}{rrr|r} 1 & 3 & 5 & 14 \\ 0 & 7 & 13 & 25 \\ 0 & 0 & 8 & 24 \end{array}\right]$$

The system is now in triangular form.

Stage 3: Back Substitution

Here we solve the equations from bottom to top. At each step of the back substitution process we encounter equations which only have a **single** unknown and so can be easily solved.

In full the equations are

$$\begin{aligned} x_1 + 3x_2 + 5x_3 &= 14 \\ 7x_2 + 13x_3 &= 25 \\ 8x_3 &= 24 \end{aligned}$$

From the last equation we see that $x_3 = 3$.

Substituting this value into the second equation gives

$$7x_2 + 39 = 25 \quad \text{or} \quad 7x_2 = -14 \quad \text{so that} \quad x_2 = -2.$$

Finally, using these values for x_2 and x_3 in equation 1 gives $x_1 - 6 + 15 = 14$. Hence $x_1 = 5$. The solution is therefore $[x_1, x_2, x_3]^T = [5, -2, 3]^T$

Example 4:
Solve the following:

$$\begin{aligned} 2x_1 - 3x_2 + 4x_3 &= 2 \\ 4x_1 + x_2 + 2x_3 &= 2 \\ x_1 - x_2 + 3x_3 &= 3 \end{aligned}$$

Augmented matrix

$$\left[\begin{array}{rrr|r} 2 & -3 & 4 & 2 \\ 4 & 1 & 2 & 2 \\ 1 & -1 & 3 & 3 \end{array}\right] \quad R1 \leftrightarrow R3 \quad \Rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 4 & 1 & 2 & 2 \\ 2 & -3 & 4 & 2 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 4 & 1 & 2 & 2 \\ 2 & -3 & 4 & 2 \end{array}\right] \quad \begin{array}{l} R2 - 4R1 \\ R3 - 2R1 \end{array} \quad \Rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 5 & -10 & -10 \\ 0 & -1 & -2 & -4 \end{array}\right]$$

$$\left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 5 & -10 & -10 \\ 0 & -1 & -2 & -4 \end{array}\right] \; R2 \div 5 \Rightarrow \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & -1 & -2 & -4 \end{array}\right] \; \underset{R3+R2}{\Rightarrow} \left[\begin{array}{rrr|r} 1 & -1 & 3 & 3 \\ 0 & 1 & -2 & -2 \\ 0 & 0 & -4 & -6 \end{array}\right]$$

The equations in full are

$$\begin{aligned} x_1 - x_2 + 3x_3 &= 3 \\ x_2 - 2x_3 &= -2 \\ -4x_3 &= -6. \end{aligned}$$

The last equation reduces to $x_3 = \frac{3}{2}$.

Using this value in the second equation gives $x_2 - 3 = -2$ so that $x_2 = 1$.

Finally, $x_1 - 1 + \frac{9}{2} = 3$ so that $x_1 = -\frac{1}{2}$.

The solution is therefore $[x_1, x_2, x_3]^T = \left[-\frac{1}{2}, 1, \frac{3}{2}\right]^T$.

Equations which have an infinite number of solutions

Consider the following system of equations

$$\begin{aligned} x_1 + x_2 - 3x_3 &= 3 \\ 2x_1 - 3x_2 + 4x_3 &= -4 \\ x_1 - x_2 + x_3 &= -1 \end{aligned}$$

In augmented form we have:

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 2 & -3 & 4 & -4 \\ 1 & -1 & 1 & -1 \end{array}\right]$$

Now performing the usual Gauss elimination operations we have

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 2 & -3 & 4 & -4 \\ 1 & -1 & 1 & -1 \end{array}\right] \begin{array}{c} \\ R2 - 2 \times R1 \\ R3 - R1 \end{array} \Rightarrow \left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & -5 & 10 & -10 \\ 0 & -2 & 4 & -4 \end{array}\right]$$

Now applying $R2 \div -5$ and $R3 \div -2$ gives

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 1 & -2 & 2 \end{array}\right]$$

Then $R2 - R3$ gives

$$\left[\begin{array}{ccc|c} 1 & 1 & -3 & 3 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

We see that all the elements in the last row are zero. This means that the variable x_3 can take any value whatsoever, so let $x_3 = t$ then using back substitution the second row now implies

We see that all the elements in the last row are zero. This means that the variable x_3 can take any value whatsoever, so let $x_3 = t$ then using back substitution the second row now implies

$$x_2 = 2 + 2x_3 = 2 + 2t$$

and then the first row implies

$$x_1 = 3 - x_2 + 3x_3 = 3 - (2 + 2t) + 3(t) = 1 + t$$

In this example the system of equations has an infinite number of solutions:

$$x_1 = 1 + t, \qquad x_2 = 2 + 2t, \qquad x_3 = t \quad \text{or} \quad [x_1, x_2, x_3]^T = [1 + t, 2 + 2t, t]^T$$

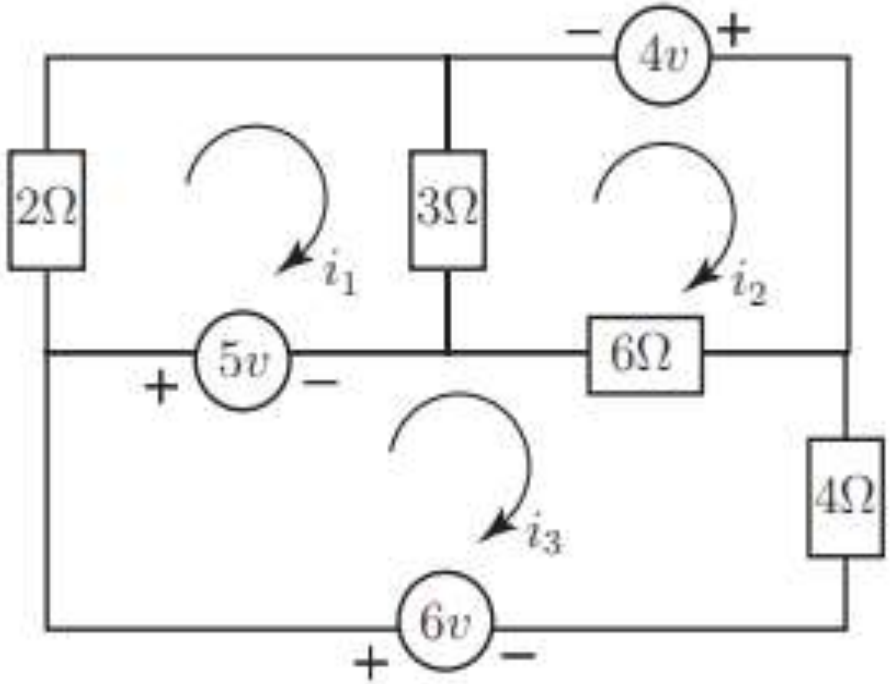
where t can be assigned any value. For every value of t these expressions for x_1, x_2 and x_3 will simultaneously satisfy each of the three given equations.

Systems of linear equations arise in the modelling of electrical circuits or networks. By breaking down a complicated system into simple loops, Kirchhoff's law can be applied. This leads to a set of linear equations in the unknown quantities (usually currents) which can easily be solved by one of the methods described in this Workbook.

APPLICATION EXAMPLES:

Currents in three loops

In the circuit shown find the currents (i_1, i_2, i_3) in the loops.



Solution

Loop 1 gives

$$2(i_1) + 3(i_1 - i_2) = 5 \quad \rightarrow \quad 5i_1 - 3i_2 = 5$$

Loop 2 gives

$$6(i_2 - i_3) + 3(i_2 - i_1) = 4 \quad \rightarrow \quad -3i_1 + 9i_2 - 6i_3 = 4$$

Loop 3 gives

$$6(i_3 - i_2) + 4(i_3) = 6 - 5 \quad \rightarrow \quad -6i_2 + 10i_3 = 1$$

Note that in loop 3, the current generated by the $6v$ cell is positive and for the $5v$ cell negative in the direction of the arrow.
In matrix form

$$\begin{bmatrix} 5 & -3 & 0 \\ -3 & 9 & -6 \\ 0 & -6 & 10 \end{bmatrix} \begin{bmatrix} i_1 \\ i_2 \\ i_3 \end{bmatrix} = \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

Solving gives

$$i_1 = \frac{34}{15}, \quad i_2 = \frac{19}{9}, \quad i_3 = \frac{41}{30}$$

DETALIS:

$5I_1$ -3I_2 - 0I_3 = 5
$-3I_1$ + 9I_2 - 6I_3 =4
$0I_1$ - 6I_2 + 10I_3 =1

$$\begin{bmatrix} 5 & -3 & 0 \\ -3 & 9 & -6 \\ 0 & -6 & 10 \end{bmatrix} \begin{bmatrix} 5 \\ 4 \\ 1 \end{bmatrix}$$

MULTIPLY R1 BY 3 AND R2 BY 5 AS FOLLOWS:

$$\begin{bmatrix} 15 & -9 & 0 \\ -15 & 45 & -30 \\ 0 & -6 & 10 \end{bmatrix} \begin{bmatrix} 15 \\ 20 \\ 1.0 \end{bmatrix}$$

R2 + R1 YIELDS:

$$\begin{bmatrix} 15 & -9 & 0 \\ 0 & 36 & -30 \\ 0 & -6 & 10 \end{bmatrix} \begin{bmatrix} 15 \\ 35 \\ 1 \end{bmatrix}$$

MULTIPLY R3 BY 6

$$\begin{bmatrix} 15 & -9 & 0 \\ 0 & 36 & -30 \\ 0 & -36 & 60 \end{bmatrix} \begin{bmatrix} 15 \\ 35 \\ 6 \end{bmatrix}$$

R3 –R2 YIELDS:

$$\begin{bmatrix} 15 & -9 & 0 \\ 0 & 36 & -30 \\ 0 & 0 & 30 \end{bmatrix} \begin{bmatrix} 15 \\ 35 \\ 41 \end{bmatrix}$$

$I_3 \frac{41}{30}$, $36I_2$ -30 * $\frac{41}{30}$ = 35 YIELDS , $I_2 = \frac{76}{36} = \frac{19}{9}$ AND $I_1 = \frac{34}{15}$

Exercises

Solve the following using Gauss elimination:

1.
$$\begin{array}{rcrcrcr} 2x_1 & + & x_2 & - & x_3 & = & 0 \\ x_1 & & & + & x_3 & = & 4 \\ x_1 & + & x_2 & + & x_3 & = & 0 \end{array}$$

2.
$$\begin{array}{rcrcrcr} x_1 & - & x_2 & + & x_3 & = & 1 \\ -x_1 & & & + & x_3 & = & 1 \\ x_1 & + & x_2 & - & x_3 & = & 0 \end{array}$$

3.
$$\begin{array}{rcrcrcr} x_1 & + & x_2 & + & x_3 & = & 2 \\ 2x_1 & + & 3x_2 & + & 4x_3 & = & 3 \\ x_1 & - & 2x_2 & - & x_3 & = & 1 \end{array}$$

4.
$$\begin{array}{rcrcrcr} x_1 & - & 2x_2 & - & 3x_3 & = & -1 \\ 3x_1 & + & x_2 & + & x_3 & = & 4 \\ 11x_1 & - & x_2 & - & 3x_3 & = & 10 \end{array}$$

Answers

(1) $x_1 = \frac{8}{3}$, $x_2 = -4$, $x_3 = \frac{4}{3}$

(2) $x_1 = \frac{1}{2}$, $x_2 = 1$, $x_3 = \frac{3}{2}$

(3) $x_1 = 2$, $x_2 = 1$, $x_3 = -1$

(4) infinite number of solutions: $x_1 = t$, $x_2 = 11 - 10t$, $x_3 = 7t - 7$