System of Linear Equations

Linear systems of equations arise in many problems in engineering and science, as well as in mathematics, such as the study of numerical solutions of boundary-value problems and partial differential equations.

This lecture deals with simultaneous linear algebraic equations that can be represented generally as

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \vdots \qquad \vdots \quad &= \vdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n &= b_n \end{aligned}$$

for the unknowns $x_1, x_2, \ldots, x_n$, given the coefficients a_{ij} , $i, j = 1, 2, \ldots, n$ and the constants b_i, $i = 1, 2, \ldots, n$.

Matrices and Matrix Operations

Before studying linear systems of equations, it is useful to consider some algebra associated with matrices.

DEFINITION

An n by m matrix is a rectangular array of real or complex numbers that can be written as

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$$\mathbf{A} = (a_{ij}) = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{bmatrix}$$

a_{ij} denotes the element or entry that occurs in row i and column j of A, and the size of a matrix is described by specifying the number of rows and columns that occur in the matrix.

Example

Let

$$A = \begin{bmatrix} -1 & 3 & 4 & 0 \\ 5 & -6 & 2 & 7 \\ 3 & 0 & 1 & 2 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -3 & 4 & 2 \end{bmatrix}, \quad C = \begin{bmatrix} \pi \\ 3 \\ 0 \end{bmatrix}.$$

In these examples, A is a 3 by 4 (written3 × 4) matrix, B has only one row and is a row vector, and C has only one column and is a column vector.

The following definition gives the basic operations on matrices.

DEFINITION

(i) If A and B are two matrices of order n×m, then the sum of A and B is the n×m matrix C = A + B whose entries are

$$c_{ij} = a_{ij} + b_{ij}$$

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(ii) *If A is a matrix of order* $n \times m$ *and* λ *a real number, then the product of* λ *and A is the* $n \times m$ *matrix* $C = \lambda A$ *whose entries are*

$$c_{ij} = \lambda\, a_{ij}$$

(iii) *If A is a matrix of order* $n \times m$ *and B is a matrix of order* $m \times p$*, then the matrix product of A and B is the* $n \times p$ *matrix* $C = AB$ *whose entries are*

$$c_{ij} = \sum_{k=1}^{m} a_{ik} b_{kj}$$

(iv) If A is a matrix of order $n \times m$*, then the transpose of A is the* $m \times n$ *matrix* $C = A^T$ *whose entries are*

$$c_{ij} = a_{ji}$$

Example

If

$$A = \begin{bmatrix} 1 & -2 & 4 \\ 3 & 5 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & 3 & -7 \\ 1 & -3 & 8 \end{bmatrix}$$

then

$$A + B = \begin{bmatrix} 1 & 1 & -3 \\ 4 & 2 & 2 \end{bmatrix}, \quad -3B = \begin{bmatrix} 0 & -9 & 21 \\ -3 & 9 & -24 \end{bmatrix},$$

$$A - B = \begin{bmatrix} 1 & -5 & 11 \\ 2 & 8 & -14 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 3 \\ -2 & 5 \\ 4 & -6 \end{bmatrix}$$

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If

$$A = \begin{bmatrix} 1 & 3 & -1 \\ 2 & 0 & -1 \\ 4 & 2 & -2 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 1 \\ -3 & 2 \\ 3 & 0 \end{bmatrix}$$

then

$$AB = \begin{bmatrix} -10 & 7 \\ 1 & 2 \\ -4 & 8 \end{bmatrix}, \quad AA = A^2 = \begin{bmatrix} 3 & 1 & -2 \\ -2 & 4 & 0 \\ 0 & 8 & -2 \end{bmatrix}.$$

Certain square matrices have special properties. For example, if the elements below the main diagonal are zero, the matrix is called an upper triangular matrix (U). Thus,

$$U = \begin{bmatrix} 1 & -2 & -1 \\ 0 & 3 & 6 \\ 0 & 0 & 2 \end{bmatrix}$$

A square matrix, in which all elements above the main diagonal are zero, is called lower triangular matrix (L)

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Naive Gaussian elimination

Three useful operations can be applied to a linear system of equations that yield an equivalent system, meaning one that has the same solutions. These operations are as follows:

(1) Swap one equation for another.

(2) Add or subtract a multiple of one equation from another.

(3) Multiply an equation by a nonzero constant.

Consider the system

Example: *Apply Gaussian elimination in tableau form for the system of three equations in three unknowns:*

$$x + 2y - z = 3$$
$$2x + y - 2z = 3$$
$$-3x + y + z = -6.$$

This is written in tableau form as

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & 3 \\ 2 & 1 & -2 & 3 \\ -3 & 1 & 1 & -6 \end{array}\right].$$

Two steps are needed to eliminate column 1:

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$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 2 & 1 & -2 & 3 \\ -3 & 1 & 1 & -6 \end{array}\right] \xrightarrow{\substack{\text{subtract } 2 \times \text{row 1} \\ \text{from row 2}}} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ -3 & 1 & 1 & -6 \end{array}\right]$$

$$\xrightarrow{\substack{\text{subtract } -3 \times \text{row 1} \\ \text{from row 3}}} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 7 & -2 & 3 \end{array}\right]$$

and one more step to eliminate column 2:

$$\left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 7 & -2 & 3 \end{array}\right] \xrightarrow{\substack{\text{subtract } -\frac{7}{3} \times \text{row 2} \\ \text{from row 3}}} \left[\begin{array}{rrr|r} 1 & 2 & -1 & 3 \\ 0 & -3 & 0 & -3 \\ 0 & 0 & -2 & -4 \end{array}\right]$$

Returning to the equations

$$x + 2y - z = 3$$

$$-3y = -3$$

$$-2z = -4,$$

The solution is x = 3, y = 1and z = 2.

Exercises

1. Use Gaussian elimination to solve the systems:

(a) $\begin{aligned} 2x - 3y &= 2 \\ 5x - 6y &= 8 \end{aligned}$ (b) $\begin{aligned} x + 2y &= -1 \\ 2x + 3y &= 1 \end{aligned}$ (c) $\begin{aligned} -x + y &= 2 \\ 3x + 4y &= 15 \end{aligned}$

2. Use Gaussian elimination to solve the systems:

(a) $\begin{aligned} 2x - 2y - z &= -2 \\ 4x + y - 2z &= 1 \\ -2x + y - z &= -3 \end{aligned}$ (b) $\begin{aligned} x + 2y - z &= 2 \\ 3y + z &= 4 \\ 2x - y + z &= 2 \end{aligned}$ (c) $\begin{aligned} 2x + y - 4z &= -7 \\ x - y + z &= -2 \\ -x + 3y - 2z &= 6 \end{aligned}$

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The LU Factorization

The LU factorization is a matrix representation of Gaussian elimination. It consists of writing the coefficient matrix A as a product of a lower triangular matrix L and an upper triangular matrix U.

DEFINITION

*An $m \times n$ matrix L is **lower triangular** if its entries satisfy $l_{ij} = 0$ for $i < j$. An $m \times n$ matrix U is **upper triangular** if its entries satisfy $u_{ij} = 0$ for $i > j$.*

Example *Find the LU factorization of*

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix}.$$

The elimination steps proceed as before:

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} \xrightarrow{\substack{\text{subtract } 2 \times \text{row } 1 \\ \text{from row } 2}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ -3 & 1 & 1 \end{bmatrix}$$

$$\xrightarrow{\substack{\text{subtract } -3 \times \text{row } 1 \\ \text{from row } 3}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 7 & -2 \end{bmatrix}$$

$$\xrightarrow{\substack{\text{subtract } -\frac{7}{3} \times \text{row } 2 \\ \text{from row } 3}} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = U.$$

The lower triangular L matrix is formed by putting 1's on the main diagonal and the multipliers in the lower triangle—in the specific places they were used for elimination. That is,

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -\frac{7}{3} & 1 \end{bmatrix}.$$

Now check that

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -\frac{7}{3} & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} = A.$$

Back substitution with the LU factorization

Once L and U are known, the problem Ax = b can be written as LUx = b. Define a new "auxiliary" vector c = Ux. Then back substitution is a two-step procedure:

(a) Solve Lc = b for c.

(b) Solve Ux = c for x.

Example *Solve system*

$$\begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}.$$

using the LU factorization

$$LU = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 3 & -4 \end{bmatrix} = A.$$

From the right-hand side of system , b = [3,2]. Step (a) is

$$\begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix}\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

which corresponds to the system

$$\begin{aligned} c_1 + 0c_2 &= 3 \\ 3c_1 + c_2 &= 2. \end{aligned}$$

Starting at the top, the solutions are $c_1 = 3, c_2 = -7$.

Step (b) is

$$\begin{bmatrix} 1 & 1 \\ 0 & -7 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 3 \\ -7 \end{bmatrix},$$

which corresponds to the system

$$\begin{aligned} x_1 + x_2 &= 3 \\ -7x_2 &= -7. \end{aligned}$$

Starting at the bottom, the solutions are $x_2 = 1, x_1 = 2$. This agrees with the “classical” Gaussian elimination computation done earlier. ◄

Example *Solve system*

$$\boldsymbol{x + 2y - z = 3}$$

$$\boldsymbol{2x + y - 2z = 3}$$

$$\boldsymbol{-3x + y + z = -6.}$$

using the LU factorization

$$\begin{bmatrix} 1 & 2 & -1 \\ 2 & 1 & -2 \\ -3 & 1 & 1 \end{bmatrix} = LU = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -\frac{7}{3} & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix}$$

From the right-hand side of system , b = (3,3, −6). The Lc = b step is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -3 & -\frac{7}{3} & 1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \\ -6 \end{bmatrix},$$

which corresponds to the system

$$\begin{aligned} c_1 &= 3 \\ 2c_1 + c_2 &= 3 \\ -3c_1 - \frac{7}{3}c_2 + c_3 &= -6. \end{aligned}$$

Starting at the top, the solutions are c1 = 3,c2 = −3,c3 = −4. The Ux = c step is

$$\begin{bmatrix} 1 & 2 & -1 \\ 0 & -3 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ -4 \end{bmatrix}$$

which corresponds to the system

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 3 \\ -3x_2 &= -3 \\ -2x_3 &= -4, \end{aligned}$$

and is solved from the bottom up to give x = [3,1,2].

Exercises

Find the LU factorization of the given matrices. Check by matrix multiplication.

$$\text{(a)}\quad \begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix} \quad \text{(b)}\quad \begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix} \quad \text{(c)}\quad \begin{bmatrix} 1 & -1 & 1 & 2 \\ 0 & 2 & 1 & 0 \\ 1 & 3 & 4 & 4 \\ 0 & 2 & 1 & -1 \end{bmatrix}$$

Solve the system by finding the LU factorization and then carrying out the two-step back substitution.

$$\text{(a)}\quad \begin{bmatrix} 3 & 1 & 2 \\ 6 & 3 & 4 \\ 3 & 1 & 5 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 3 \end{bmatrix} \quad \text{(b)}\quad \begin{bmatrix} 4 & 2 & 0 \\ 4 & 4 & 2 \\ 2 & 2 & 3 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

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