ORDINARY DIFFERENTIAL EQUATION

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1. FIRST ORDER D. E.

SOLUTION OF FIRST ORDER DIFFERENTIAL EQUATIONS

by separation of Variables

* Equations which can be written in the form:

$\frac{dy}{dx} = f(x)$; $\frac{dy}{dx} = f(y)$; $\frac{dy}{dx} = f(x) \cdot f(y)$

Can all be solved by Integration. In each case it is possible to separate the "y" to one side of the equation and the "x" to the other, Solving such equations is therefore known as solution by "Separation of Variables".

Ex. 1 Determine the general solution of

$$x \frac{dy}{dx} = 2 - 4x^3$$

Rearranging the above equation gives;

$$\frac{dy}{dx} = \frac{2 - 4x^3}{x} \Rightarrow \frac{dy}{dx} = \frac{2}{x} - 4x^2$$

$$\int dy = \int \left(\frac{2}{x} - 4x^2 \right) dx$$

$$y = \int \left(\frac{2}{x} - 4x^2 \right) dx = 2\int \frac{dx}{x} - 4\int x^2 dx$$

$$y = 2\ln x - \frac{4}{3}x^3 + c$$

which is the general solution.

<u>Ex. 2</u> Determine the Particular Solution of the differential equation $5\frac{dy}{dx} + 2x = 3$; giving the boundary Conditions (B.Cs.) ; $y = 1\frac{2}{5}$ when $x = 2$

since; $5\frac{dy}{dx} + 2x = 3$

Then; $\frac{dy}{dx} = \frac{3 - 2x}{5} = \frac{3}{5} - \frac{2}{5}x$

$$\int dy = \int \left(\frac{3}{5} - \frac{2}{5}x \right) dx$$

$$y = \frac{3}{5}x - \frac{1}{5}x^2 + c$$

which is the general solution

Substituting the boundary Conditions $y = 1\frac{2}{5}$; $x = 2$ to evaluate "c" gives;

$$1\frac{2}{5} = \frac{6}{5} - \frac{4}{5} + c \Rightarrow c = 1$$

∴ The Particular Solution : $y = \frac{3}{5}x - \frac{1}{5}x^2 + 1$.

<u>Ex 3</u> Determine the Particular Solution of

$(y^2-1)\dfrac{dy}{dx} = 3y$ given that $y = 1$ when $x = 2\frac{1}{6}$

Rearranging gives;

$$\frac{(y^2-1)}{3y}\,dy = dx \Rightarrow \int dx = \int\left(\frac{y}{3} - \frac{1}{3y}\right)dy$$

Integrating gives;

$$\int dx = \int \frac{y}{3}\,dy - \frac{1}{3}\int \frac{dy}{y}$$

i.e. $x = \dfrac{y^2}{6} - \dfrac{1}{3}\ln y + c$

which is the general solution.

When; $y = 1$; $x = 2\frac{1}{6}$

Thus: $2\frac{1}{6} = \frac{1}{6} - \frac{1}{3}\ln 1 + c \Rightarrow c = 2$

$\therefore$ The Particular Solution is:

$$x = \frac{y^2}{6} - \frac{1}{3}\ln y + 2$$

Jage (3)

Ex-4. Solve the equation: $4xy\frac{dy}{dx} = y^2 - 1$

Separating the variables gives:

$$\left(\frac{4y}{y^2-1}\right)dy = \frac{1}{x}dx$$

Integrating both sides:

$$\int\left(\frac{4y}{y^2-1}\right)dy = \int\frac{dx}{x}$$

Using the substitution $u = y^2 - 1$:

$$du = 2y\,dy \Rightarrow y\,dy = \frac{du}{2}$$

$$4\int\left(\frac{y\,dy}{y^2-1}\right) = \int\frac{dx}{x}$$

$$4\int\frac{du}{2u} = \int\frac{dx}{x}$$

$$2\int\frac{du}{u} = \int\frac{dx}{x} \Rightarrow 2\ln u = \ln x + c$$

$$2\ln(y^2-1) = \ln x + C \quad \text{---} (1)$$

This is the first valid general solution

$$2\ln(y^2-1) = \ln x + c$$

$$\ln(y^2-1)^2 - \ln x = c$$

$$\ln\left[\frac{(y^2-1)^2}{x}\right] = c$$

$$\frac{(y^2-1)^2}{x} = e^c \quad \text{---} (2)$$

This is the second valid solution.

If in eq-(1); $c = \ln A$

$$\ln(y^2-1)^2 = \ln x + \ln A$$

$$\ln(y^2-1)^2 = \ln Ax$$

$$(y^2-1)^2 = Ax \quad \text{---} (3)$$

In this way, there are three valid Solutions for the ODE:

$$4xy\frac{dy}{dx} = y^2 - 1$$

Ex. 5. For an adiabatic expansion of a gas

$$C_v \frac{dP}{P} + C_P \frac{d\forall}{\forall} = 0$$

Where, C_p and C_v are constants, given

$$\gamma = \frac{C_P}{C_v}$$

show that $P V^{\gamma} = C$

(Thermodynamics Problem)

Separating the variables gives:

$$C_v \frac{dP}{P} = - C_P \frac{d\forall}{\forall}$$

Integrating both sides; gives:—

$$C_v \int \frac{dP}{P} = - C_P \int \frac{d\forall}{\forall}$$

$$C_v \ln P = - C_P \ln \forall + k$$

Dividing both sides by "C_v"

$$\ln P = - \frac{C_P}{C_v} \ln \forall + \frac{k}{C_v} \quad ; \quad K = \frac{k}{C_v}$$

$$\ln P = -\gamma \ln \forall + K \Rightarrow \ln P + \gamma \ln \forall = K$$

$$\ln P \forall^{\gamma} = K \Rightarrow P \forall^{\gamma} = e^{K} \Rightarrow P \forall^{\gamma} = C$$

Tutorial on Separation of Variable Method for Solving first Order differential equations.

Ex. Solve the following differential equation:

1. $x(2y-3)\,dx = -(x^2+1)\,dy$

$$-\int \frac{x}{(x^2+1)}\,dx = \int \frac{dy}{(2y-3)}$$

$$-\frac{1}{2}\int \frac{2x}{(x^2+1)}\,dx = \frac{1}{2}\int \frac{2\,dy}{(2y-3)}$$

$$-\frac{1}{2}\ln(x^2+1) = \frac{1}{2}\ln(2y-3) + c$$

2. $x^2(y^2+1)\,dx + y\sqrt{x^3+1}\,dy = 0$

$$x^2(y^2+1)\,dx = -y\sqrt{x^3+1}\,dy$$

$$\frac{x^2}{\sqrt{x^3+1}}\,dx = -\frac{y}{(y^2+1)}\,dy$$

$$\int x^2(x^3+1)^{-1/2}\,dx = \int \frac{-y}{(y^2+1)}\,dy$$

$$\frac{1}{3}\int (x^3+1)^{-1/2} 3x^2\,dx = -\frac{1}{2}\int \frac{2y\,dy}{(y^2+1)}$$

$$\frac{1}{3}\,\frac{(x^3+1)^{1/2}}{\frac{1}{2}} = -\frac{1}{2}\ln(y^2+1) + c$$

$$\frac{2}{3}\sqrt{x^3+1} + \frac{1}{2}\ln(y^2+1) = c$$

3. $\frac{dy}{dx} = \frac{e^x}{e^y}$

$\int e^y\,dy = \int e^x\,dx \rightarrow e^y - e^x = c$

4. $\frac{dy}{dx}\sin x + \cosh 2y = 0$

$\sin x \frac{dy}{dx} = -\cosh 2y$

$$-\frac{dy}{\cosh 2y} = \frac{dx}{\sin x}$$

$-\int \operatorname{sech} 2y\,dy = \int \csc x\,dx$

$\frac{1}{2}\ln(\operatorname{sech} 2y + \tanh 2y) - \ln|\csc x + \cot x| = c$

5. $\frac{dy}{dx} = \frac{x}{y} \ln x$

$\int y \, dy = \int x \ln x \, dx$

The L.H.S. can be integrated using Int. by Parts

$\int u \, dv = uv - \int v \, du$

let $u = \ln x \rightarrow du = \frac{1}{x} dx$

$dv = x \, dx \rightarrow v = \frac{x^2}{2}$

$\int x \ln x \, dx = \ln x \cdot \frac{x^2}{2} - \int \frac{x^2}{2} \frac{1}{x} dx$

$= \frac{x^2}{2} \ln x - \frac{1}{2} \int x \, dx$

$\therefore \int x \ln x \, dx = \frac{x^2}{2} \ln x - \frac{x^4}{4} + c$

$\therefore \int y \, dy = \frac{x^2}{2} \ln x - \frac{x^4}{4} + c$

$\frac{y^2}{2} - \frac{x^2}{2} \ln x + \frac{x^4}{4} = c$

The general Solution

6. $x e^{2y} dy + \frac{x^{\cdot}}{1} \cdot dx = 0$

$$x e^{2y} dy = -\frac{x^2+1}{y} dx$$

$$e^{2y} y \, dy = -\frac{x^2+1}{x} dx$$

$$\int e^{2y} y \, dy = \int -\left(x + \frac{1}{x}\right) dx$$

R.H.S. Can be integrated by using Int. by Parts

let $u = y$; $du = dy$

$dv = e^{2y} dy \rightarrow v = \frac{1}{2} e^{2y}$

$$\int u \cdot dv = uv - \int v \cdot du$$

$$\int e^{2y} y \, dy = y \frac{1}{2} e^{2y} - \int \frac{1}{2} e^{2y} dy$$

$$= \frac{1}{2} y e^{2y} - \frac{1}{4} e^{2y} + C$$

$$\therefore \frac{1}{2} y e^{2y} - \frac{1}{4} e^{2y} = -\left(\frac{x^2}{2} + \ln x\right) + C$$

$$\frac{1}{2} y e^{2y} - \frac{1}{4} e^{2y} + \frac{x^2}{2} + \ln x = C$$

The general Solution.

1.2. Homogeneous first Order differential equations.

- An equation of the form $P\frac{dy}{dx} = Q$, where P; Q are functions of both "x" and "y" of the same degree throughout is said to be homogeneous in y and x.
- For example; $f(x,y) = x^2+3xy+y^2$ is a homogenous function since each of the three terms are of degree 2. However, $f(x,y) = \frac{x^2-y}{2x^2+y^2}$ is not homogeneous since the term in "y" in the numerator is of degree "1" and the other terms are of degree "2".

Procedure for Solving Homogeneous differential equations in the form of $P\frac{dy}{dx} = Q$.

(i) Rearrange $P\frac{dy}{dx} = Q$ into the form $\frac{dy}{dx} = \frac{Q}{P}$

(ii) Make the substitution "$y = vx$ (where "v" is a func of "x"), from which, $\frac{dy}{dx} = v(1) + x\frac{dv}{dx}$, by the product rule.

(iii) sub for both "y" and "$\frac{dy}{dx}$" in the equation: $\frac{dy}{dx} = \frac{Q}{P}$

(iv) Solve it using separation of variable as explained in section (1.1.)

(v) sub $v = \frac{y}{x}$ to solve in terms of original variables.

Ex.1. Solve the differential equation:

$$y - x = x\frac{dy}{dx} \quad ; \text{given } x = 1 \text{ when } y = 2.$$

$y - x = x\dfrac{dy}{dx}$ gives:

$$\frac{dy}{dx} = \frac{y-x}{x} \quad ; \text{which is homogeneous in } x \text{ and } y.$$

let $y = vx \rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{vx - x}{x} = \frac{x(v-1)}{x} = v - 1$$

$$v + x\frac{dv}{dx} = v - 1 \rightarrow x\frac{dv}{dx} = v - 1 - v$$

$$x\frac{dv}{dx} = -1 \rightarrow \int dv = \int -\frac{dx}{x}$$

$$v = -\ln x + c$$

Replacing "v" by $\dfrac{y}{x}$

gives; $\dfrac{y}{x} = -\ln x + c$

The general solution.

when; $x = 1$; $y = 2 \rightarrow c = 2$

$\therefore$ The Particular solution is: $\dfrac{y}{x} = -\ln x + 2$

$$y = -x(\ln x - 2)$$

$$y = x(2 - \ln x)$$

Ex. 2. Find the Particular solution of the equation:

$$x \frac{dy}{dx} = \frac{x^2+y^2}{y}$$; given the boundary conditions that

$y = 4$ and $x = 1$

Rearranging $x \frac{dy}{dx} = \frac{x^2+y^2}{y}$ gives:

$$\frac{dy}{dx} = \frac{x^2+y^2}{xy}$$ (Homogeneous equation of degree "2")

let $vx = y \Rightarrow \frac{dy}{dx} = v + x \frac{dv}{dx}$

$$v + x \frac{dv}{dx} = \frac{x^2 + v^2x^2}{x(vx)} = \frac{\cancel{x^2}(1+v^2)}{\cancel{x^2}\, v} = \frac{1+v^2}{v}$$

$$v + x \frac{dv}{dx} = \frac{1+v^2}{v}$$

Separating the variables gives:

$$x \frac{dv}{dx} = \frac{1+v^2}{v} - v = \frac{1+v^2 - v^2}{v} = \frac{1}{v}$$

$$x \frac{dv}{dx} = \frac{1}{v} \Rightarrow \int v \, dv = \int \frac{dx}{x}$$

$$\frac{v^2}{2} = \ln x + c \quad \therefore v = \frac{y}{x}$$

$\frac{y^2}{2x^2} = \ln x + c$, which is the general Solution when $x=1;\ y=4 \Rightarrow c=8$

Hence; the Particular Solution is,

$$\frac{y^2}{2x^2} = \ln x + 8$$

$$y^2 = 2x^2(8+\ln x)$$

Ex.3. Solve the equation:

$7x(x-y)\,dy = 2(x^2+6xy-5y^2)\,dx$

given that $x=1$ when $y=0$

$$\frac{dy}{dx} = \frac{2x^2+12xy-10y^2}{7x^2-7xy}$$

(Which is homogeneous of degree "2")

let $y = vx \rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{2x^2+12x(vx)-10v^2x^2}{7x^2-7x(vx)}$$

$$v + x\frac{dv}{dx} = \frac{2x^2+12vx^2-10v^2x^2}{7x^2-7vx^2}$$

$$v + x\frac{dv}{dx} = \frac{2+12v-10v^2}{7-7v}$$

Separating the variables:

$$x\frac{dv}{dx} = \frac{2+12v-10v^2}{7-7v} - v$$

$$= \frac{2+12v-10v^2 - v(7-7v)}{7-7v}$$

$$= \frac{2+12v-10v^2-7v+7v^2}{7-7v}$$

$$x\frac{dv}{dx} = \frac{2+5v-3v^2}{7-7v}$$

$$\int \frac{7-7v}{2+5v-3v^2}\,dv = \int \frac{dx}{x}$$

let us integrate the L.H.S. using Partial fraction Method:

$$\int \frac{7-7v}{2+5v-3v^2}\,dv = \int \frac{7-7v}{(1+3v)(2-v)}\,dv$$

Problem (15)

$$\frac{7-7v}{(1+3v)(2-v)} = \frac{A}{(1+3v)} + \frac{B}{(2-v)} \rightarrow 7-7v = A(2-v) + B(1+3v)$$

At $v = 2 \rightarrow 7 - 7(2) = B(1+6)$

$$-7 = 7B \Rightarrow B = -1$$

At $v = -\frac{1}{3} \rightarrow 7 + \frac{7}{3} = A\left(2 + \frac{1}{3}\right)$

$$\frac{28}{3} = \frac{7}{3}A \rightarrow A = 4$$

$$\therefore \int \left(\frac{7-7v}{2+5v-3v^2}\right) dv = \int \left(\frac{4}{1+3v} - \frac{1}{2-v}\right) dv$$

$$\therefore \int \left(\frac{4}{1+3v} - \frac{1}{2-v}\right) dv = \int \frac{dx}{x}$$

$$\frac{4}{3}\int \frac{3}{(1+3v)} dv - \int \frac{dv}{2-v} = \int \frac{dx}{x}$$

$$\frac{4}{3}\ln(1+3v) + \ln(2-v) = \ln x + c$$

Replacing v by $\frac{y}{x}$

$$\frac{4}{3}\ln\left(1 + \frac{3y}{x}\right) + \ln\left(2 - \frac{y}{x}\right) = \ln x + c$$

When $x = 1, y = 0 \rightarrow c = \ln 2$

$\therefore$ Particular Sol: $\frac{4}{3}\ln \frac{x+3y}{x} + \ln \frac{2x-y}{x} = \ln x + \ln 2$

Ex.4 Show that the solution of the differential equation $x^2 - 3y^2 + 2xy\dfrac{dy}{dx} = 0$ is $y = x\sqrt{8x+1}$; given that $y=3$ when $x=1$.

Rearranging gives:

$$2xy\frac{dy}{dx} = 3y^2 - x^2 \Rightarrow \frac{dy}{dx} = \frac{3y^2 - x^2}{2xy}$$

let $y = vx$; then $\dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$

$$v + x\frac{dv}{dx} = \frac{3v^2x^2 - x^2}{2x(vx)} = \frac{x^2(3v^2-1)}{2x^2 v}$$

$$v + x\frac{dv}{dx} = \cancel{\frac{3(v^2-1)}{2v}} \quad \frac{3v^2-1}{2v}$$

$$v + x\frac{dv}{dx} = \frac{3v^2-1}{2v}$$

Separating the variables ; gives:

$$x\frac{dv}{dx} = \frac{3v^2-1}{2v} - v = \frac{3v^2-1-2v^2}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2-1}{2v}$$

$$x\frac{dv}{dx} = \frac{v^2-1}{2v} \Rightarrow \int \frac{2v}{v^2-1}\,dv = \int \frac{dx}{x}$$

$$\ln|v^2-1| = \ln|x| + c$$

Replacing v by $\frac{y}{x}$

$$\ln\left(\frac{y^2}{x^2}-1\right) = \ln x + c$$

which is the general solution.

When; $y=3$, $x=1 \rightarrow c = \ln 8$

Hence, the particular solution is:

$$\ln\left(\frac{y^2}{x^2}-1\right) = \ln x + \ln 8$$

$$\ln\left(\frac{y^2}{x^2}-1\right) = \ln 8x$$

$$\frac{y^2}{x^2} - 1 = 8x$$

$$\frac{y^2}{x^2} = 8x + 1 \rightarrow y^2 = x^2[8x+1]$$

$$\therefore y = x\sqrt{(8x+1)}$$

1.3. Linear first order differental equations

- An equation of the form $\frac{dy}{dx} + Py = Q$; where "P" and "Q" are functions of "x" only is called a linear differential equation since "y" and its derivatives are of first degree.

- Procedure to solve differential equations of the form $\frac{dy}{dx} + Py = Q$:

(I) Rearrange the differential equation in the form of $\frac{dy}{dx} + Py = Q$; Where "P and Q" are functions of "x".

(ii) Determine $\int P\,dx$

(iii) Determine the integrating factor $e^{\int P\,dx}$

(IV) Substitute $e^{\int P\,dx}$ into equation given below

$$y\,e^{\int P\,dx} = \int e^{\int P\,dx}\,Q\,dx$$

(V) Integrate the R.H.S. of the equation above to give the general solution of the differential equation. Given B.Cs., the particular solution may be determined.