Stability analysis and Rouths stability Criterion

Poles and zeros of a transfer function (T. F.):

Poles: The values of **s** that make the T.F. infinite (∞).

Note: Poles are indicated by 'X' (cross) in s-plane.

Zeros: The values of **s** that make the T.F. zero.

Note: The zeros are indicated by small circle or zero 'O' in the s-plane.

Example 1: Find the poles and zeros of the following transfer function and then plot (draw, map) the poles and zero on the s-plan.

$$G(s) = \frac{N(s)}{D(s)} = \frac{2(s+2)}{s^2+4s}$$

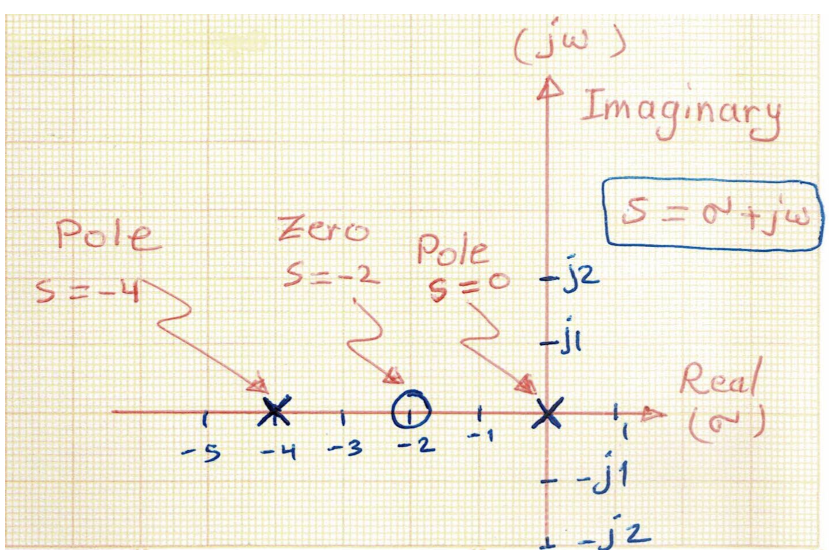
The poles can be found from

$$Denominator \;=\; D(s) = \; s^2 + 4s = s(s+4) = 0$$

Either **s=0** or s+4 = 0 ► **s = -4** *(pole1 at s = 0, pole2 at s = -4)*

The Zeros can be found from

$Numerator \;= N(s) = 2(s+2) = 0$, **s = -2** *(zero at s =-2)*



Example 2: Find the poles and zeros of the following transfer function and then plot (draw, map) the poles and zero on the s-plan

$$T(s) = G(s) = \frac{N(s)}{D(s)} = \frac{K(s+3)}{(s+4)^2(s+1)(s^2+2s+2)} \qquad \text{where K constant value}$$

The poles are the roots of the equation (characteristic equation)

$D(s) = (s+4)^2(s+1)(s^2+2s+2) = 0$

Either $s+1 = 0$, **s= -1** or $(s+4)^2 = 0$, **s=-4 , s=-4** , or $s^2+2s+2 = 0$, **s=-1+j1 , s=-1-j1**

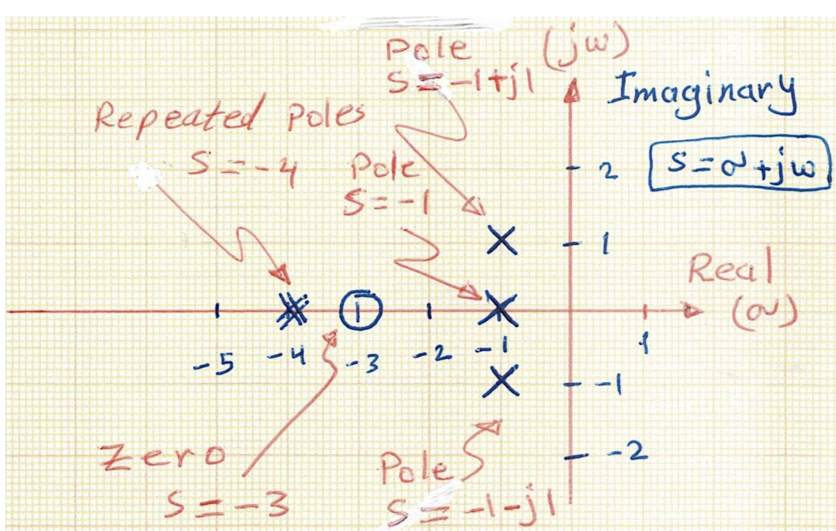
There are five poles, (s= -1, s= -4, s= -4, s= -1±j1)

So T(s) has simple pole at	s = –1,
Repeated pole at	s = –4, (two poles)
Complex conjugate poles at	s = –1 ± j1

The Zeros can be found from

$Numerator = N(s) = K(s+3) = 0$, **s = -3** *(zero at s=-3)*

It has one zero (s= -3)



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Example 3: For the following transfer function plot (draw, map) the poles and zero on the s-plan

$$\frac{C(s)}{R(s)} = \frac{(s+2)}{s\left[s^2+2s+2\right]\left[s^2+7s+12\right]}$$

The characteristic equation is,

$$s(s^2+2s+2)(s^2+7s+12) = 0 \quad \text{i.e.} \quad s(s^2+2s+2)\ (s+3)(s+4) = 0$$

i.e. System is 5th order and there are 5 poles. Poles are 0, $-1 \pm j$, –3, –4 while zero is located at '–2'.

zero at s = -2

poles:

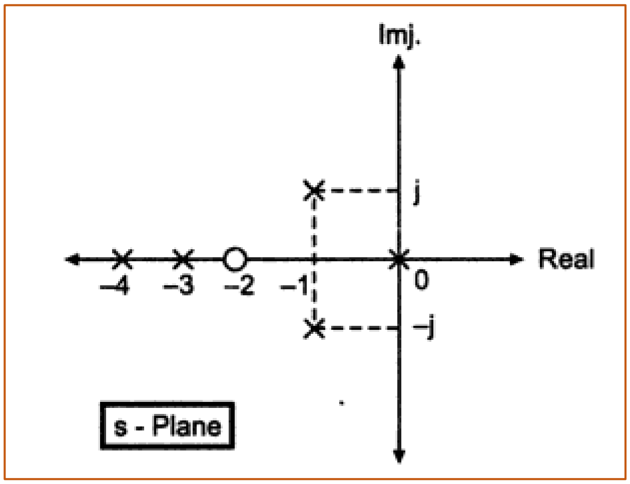
s = 0

s = -3

s = -4

s = -1 + j1

s = -1 – j1



Example 4:

A system has a pair of complex conjugate poles p1, p2 = −1 ± j2, a single real zero **z = −4**, and a gain factor K = 3. Find the transfer function of the system

Solution: The transfer function is

$$\begin{aligned} H(s) &= K\frac{s-z}{(s-p_1)(s-p_2)} \\ &= 3\frac{s-(-4)}{(s-(-1+j2))(s-(-1-j2))} \\ &= 3\frac{(s+4)}{s^2+2s+5} \end{aligned}$$

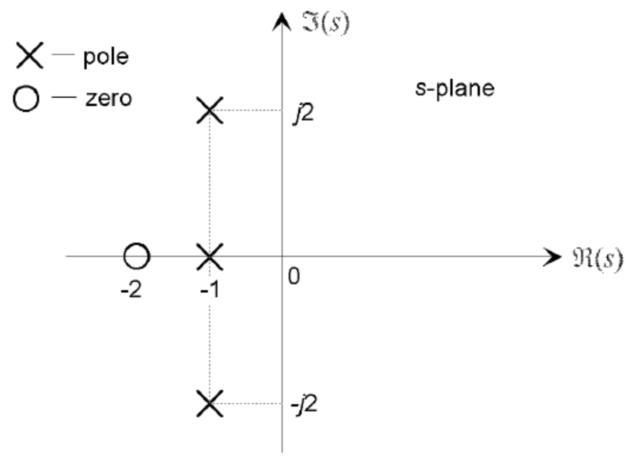
Example 5: Determine the transfer function for the following s-plain plot

Solution

There is one zero at z = -2

There are three poles at P1= -1

P2 =-1+j2 , P3 = -1-j2



The transfer is $G(s) = \frac{K(s-z)}{(s-P1)(s-P2)(s-p3)}$

$$G(s) = \frac{K(s-(-2))}{(s-(-1))(s-(-1+j2))(s-(-1-j2)}$$

$$G(s) = \frac{K(s+2)}{(s+1)(s+1-j2)(s+1+j2)}$$

$$G(s) = \frac{K(s+2)}{(s+1)(s^2+2s+5)}$$

Example 6: Find the zeros of the following transfer function

$$T(s) = \frac{2(s+1)^2 (s+2)(s^2+2s+2)}{s^3 (s+4)(s^2+6s+25)}$$

This transfer function has zeros which are roots of the equation,

$$2(s+1)^2 (s+2)(s^2+2s+2) = 0$$

i.e. Simple zero at $s = -2$

Repeated zero at $s = -1$ (twice)

Complex conjugate zeros at $s = -1 \pm j1$.

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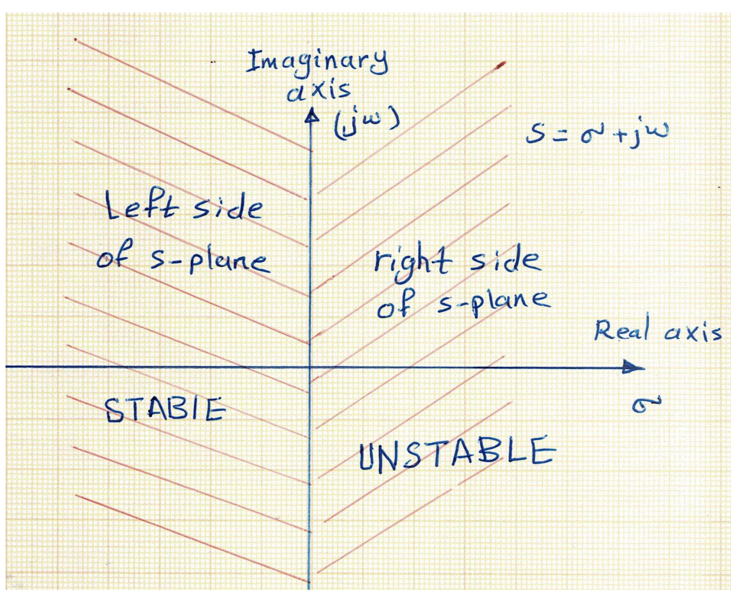
Stability of Control Systems

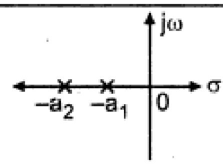
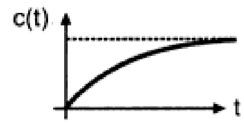
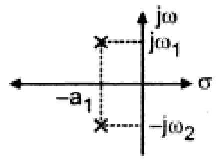
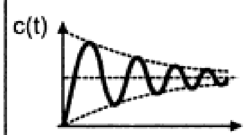
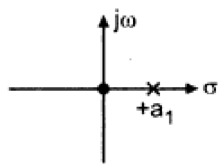
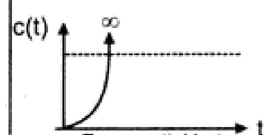
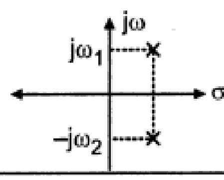
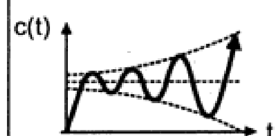
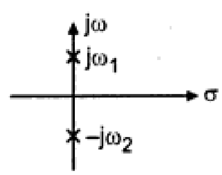
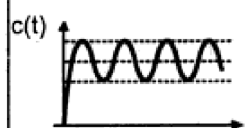
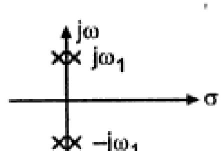
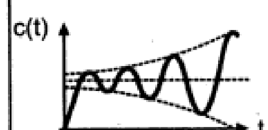
The analysis of weather the given system can pass through the transient and reach steady state successfully is called **stability analysis of the system**.

Control systems can be:

1. Absolutely stable.
2. Unstable.
3. Conditionally stable.
4. Critically or marginally stable.

- The stability of the system depends only on the location of poles but not on the location of zeros
- If the poles are located on the left side of the s-plane, then the system is stable.
- If any single pole is located on the right side of the s-plane then the system is unstable.
- If one pair of poles on the imaginary axis then the system is marginally stable.
- If two or more poles at the origin, then the system is unstable.



Sr. No.	Nature of closed loop poles	Locations of closed loop poles in s-plane	Step response	Stability condition
1.	Real, negative i.e. in L.H.S. of s-plane		 Pure exponential	Absolutely stable
2.	Complex conjugate with negative real part i.e. in L.H.S. of s-plane		 Damped oscillations	Absolutely stable
3.	Real, positive i.e. in R.H.S. of s-plane (Any one closed loop pole in right half irrespective of number of poles in left half of s-plane)		 Exponential but increasing towards ∞	Unstable
4.	Complex conjugate with positive real part i.e. in R.H.S. of s-plane		 Oscillations with increasing amplitude	Unstable
5.	Nonrepeated pair on imaginary axis without any pole in R.H.S. of s-plane			Marginally or critically stable
6.	Repeated pair on imaginary axis without any pole in R.H.S. of s-plane		 Oscillations of increasing amplitude	Unstable

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Routh's Stability Criterion

Routh's stability criterion tells us whether or not there are unstable roots in a polynomial equation without actually solving for them. This stability criterion applies to polynomials with only a finite number of terms. When the criterion is applied to a control system, information about absolute stability can be obtained directly from the coefficients of the characteristic equation.

The procedure in Routh's stability criterion is as follows:

1. Write the polynomial in s in the following form:

$$a_0 s^n + a_1 s^{n-1} + \cdots + a_{n-1} s + a_n = 0$$

2. If any of the coefficients are zero or negative in the presence of at least one positive coefficient, a root or roots exist that are imaginary or that have positive real parts. Therefore, in such a case, the system is not stable. If we are interested in only the absolute stability, there is no need to follow the procedure further.

3. The necessary but not sufficient condition for stability is that the coefficients of Equation in point 1 all be present and all have a positive sign. (If all a's are negative, they can be made positive by multiplying both sides of the equation by -1.)

4. If all coefficients are positive, arrange the coefficients of the polynomial in rows and columns according to the following pattern:

$$\begin{array}{lccccc}
s^n & a_0 & a_2 & a_4 & a_6 & \cdots \\
s^{n-1} & a_1 & a_3 & a_5 & a_7 & \cdots \\
s^{n-2} & b_1 & b_2 & b_3 & b_4 & \cdots \\
s^{n-3} & c_1 & c_2 & c_3 & c_4 & \cdots \\
s^{n-4} & d_1 & d_2 & d_3 & d_4 & \cdots \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
\cdot & \cdot & \cdot & & & \\
s^2 & e_1 & e_2 & & & \\
s^1 & f_1 & & & & \\
s^0 & g_1 & & & &
\end{array}$$

The process of forming rows continues until we run out of elements. (The total number of rows is n+1.) The coefficients b_1, b_2, b_3, and so on, are evaluated as follows:

$$b_1 = \frac{a_1 a_2 - a_0 a_3}{a_1} \qquad b_2 = \frac{a_1 a_4 - a_0 a_5}{a_1} \qquad b_3 = \frac{a_1 a_6 - a_0 a_7}{a_1} \qquad \cdot \quad \cdot \quad \cdot$$

The evaluation of the b's is continued until the remaining ones are all zero. The same pattern of cross-multiplying the coefficients of the two previous rows is followed in evaluating the c's, d's, e's, and so on. That is,

$$c_1 = \frac{b_1 a_3 - a_1 b_2}{b_1} \qquad c_2 = \frac{b_1 a_5 - a_1 b_3}{b_1} \qquad c_3 = \frac{b_1 a_7 - a_1 b_4}{b_1} \qquad \cdot \quad \cdot \quad \cdot$$

Routh's stability criterion states that the number of roots of Equation in point 1 with positive real parts is equal to the number of changes in sign of the coefficients of the first column of the array. It should be noted that the exact values of the terms in the first column need not be known; instead, only the signs are needed.

The necessary and sufficient condition that all roots of Equation in point 1 lie in the left-half s plane is that all the coefficients of Equation in point 1 be positive and all terms in the first column of the array have positive signs.

EXAMPLE 1: Let us apply Routh's stability criterion to the following third-order polynomial:

$$a_0 s^3 + a_1 s^2 + a_2 s + a_3 = 0$$

where all the coefficients are positive numbers. The array of coefficients becomes

$$\begin{array}{lcc} s^3 & a_0 & a_2 \\ s^2 & a_1 & a_3 \\ s^1 & \dfrac{a_1 a_2 - a_0 a_3}{a_1} & \\ s^0 & a_3 & \end{array}$$

The condition that all roots have negative real parts is given by

$$a_1 a_2 > a_0 a_3$$

EXAMPLE 2: Consider the following polynomial:

$$s^4 + 2s^3 + 3s^2 + 4s + 5 = 0$$

Note that in developing the array an entire row may be divided or multiplied by a positive number in order to simplify the subsequent numerical calculation without altering the stability conclusion.

$$\begin{array}{lrrr} s^4 & 1 & 3 & 5 \\ s^3 & 2 & 4 & 0 \\ & & & \\ s^2 & 1 & 5 & \\ s^1 & -6 & & \\ s^0 & 5 & & \end{array} \quad \Bigg\Vert \quad \begin{array}{lrrr} s^4 & 1 & 3 & 5 \\ s^3 & \not{2} & \not{4} & \not{0} \\ & 1 & 2 & 0 \\ s^2 & 1 & 5 & \\ s^1 & -3 & & \\ s^0 & 5 & & \end{array}$$

The second row is divided by 2.

In this example, the number of changes in sign of the coefficients in the first column **is 2.** This means that there **are two roots with positive real parts.**

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Special Case 1: If a first-column term in any row is zero, but the remaining terms are not zero or there is no remaining term, then the zero term is replaced by a very small positive number (**ϵ**) and the rest of the array is evaluated.

Example: Consider the following equation:

$$s^3 + 2s^2 + s + 2 = 0$$

The array of coefficients is

$$\begin{array}{lcc} s^3 & 1 & 1 \\ s^2 & 2 & 2 \\ s^1 & 0 \approx \epsilon & \\ s^0 & 2 & \end{array}$$

If the sign of the coefficient above the zero (ϵ) is the same as that below it, it indicates that there are a pair of imaginary roots.
Actually the equation has two roots at s = ± j.

Example: For the equation

$$s^3 - 3s + 2 = (s - 1)^2(s + 2) = 0$$

the array of coefficients is

$$\begin{array}{llcc} & s^3 & 1 & -3 \\ \text{One sign change:} & s^2 & 0 \approx \epsilon & 2 \\ & s^1 & -3 - \dfrac{2}{\epsilon} & \\ \text{One sign change:} & s^0 & 2 & \end{array}$$

There are two sign changes of the coefficients in the first column. So there are two roots in the right-half s plane. This agrees with the correct result indicated by the factored form of the polynomial equation.

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Special Case 2: If all the coefficients in any derived row are zero, it indicates that there are roots of equal magnitude lying radially opposite in the s plane—that is, two real roots with equal magnitudes and opposite signs and/or two conjugate imaginary roots.

Example: Consider the following equation:

$$s^5 + 2s^4 + 24s^3 + 48s^2 - 25s - 50 = 0$$

The array of coefficients is

s^5	1	24	-25	
s^4	2	48	-50	← Auxiliary polynomial $P(s)$
s^3	0	0		

The terms in the s^3 row are all zero. (Note that such a case occurs only in an odd-numbered row.) The auxiliary polynomial is then formed from the coefficients of the s^4 row. The auxiliary polynomial $P(s)$ is

$$P(s) = 2s^4 + 48s^2 - 50$$

which indicates that there are two pairs of roots of equal magnitude and opposite sign (that is, two real roots with the same magnitude but opposite signs or two complex-conjugate roots on the imaginary axis). These pairs are obtained by solving the auxiliary polynomial equation $P(s) = 0$. The derivative of $P(s)$ with respect to s is

$$\frac{dP(s)}{ds} = 8s^3 + 96s$$

The terms in the s^3 row are replaced by the coefficients of the last equation—that is, 8 and 96. The array of coefficients then becomes

s^5	1	24	-25	
s^4	2	48	-50	
s^3	8	96		← Coefficients of $dP(s)/ds$
s^2	24	-50		
s^1	112.7	0		
s^0	-50			

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We see that there is one change in sign in the first column of the new array. Thus, the original equation has one root with a positive real part. By solving for roots of the auxiliary polynomial equation,

$$2s^4 + 48s^2 - 50 = 0$$

we obtain

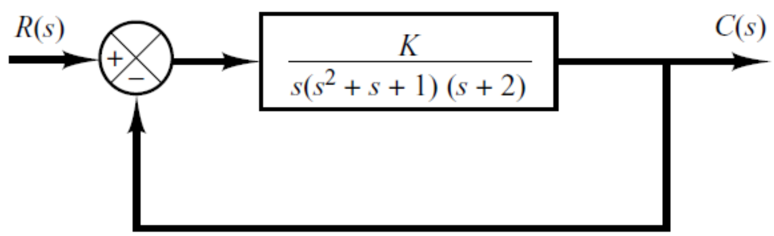
$$s^2 = 1, \qquad s^2 = -25$$

or

$$s = \pm 1, \qquad s = \pm j5$$

Application of Routh's Stability Criterion to Control-System Analysis.

Consider the system shown in Figure below. Let us determine the range of K for stability.



The closed- loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{K}{s(s^2 + s + 1)(s + 2) + K}$$

The characteristic equation is

$$s^4 + 3s^3 + 3s^2 + 2s + K = 0$$

The array of coefficients becomes

s^4	1	3	K
s^3	3	2	0
s^2	$\frac{7}{3}$	K	
s^1	$2 - \frac{9}{7}K$		
s^0	K		

For stability, K must be positive, and all coefficients in the first column must be positive. Therefore,

$$\frac{14}{9} > K > 0$$

When $K = \frac{14}{9}$, the system becomes oscillatory and, mathematically, the oscillation is sustained at constant amplitude.

أمثله متنوعة (Various examples)

Example : Consider the following equation:

$$s^3 + 6s^2 + 11s + 6 = 0$$

Solution : $a_0 = 1, \quad a_1 = 6, \quad a_2 = 11, \quad a_3 = 6, \quad n = 3$

s^3	1	11
s^2	6	6
s^1	$\frac{11 \times 6 - 6}{6} = 10$	0
s^0	6	

As there is no sign change in first column, **system is stable.**

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Example

Consider the following equation:

$$s^5 + 2s^4 + 3s^3 + 6s^2 + 2s + 1 = 0$$

s^5	1	3	2	
s^4	2	6	1	
s^3	$\boxed{0}$	1.5	0	Special Case 1
s^2	∞	...	...	Routh' array failed

Following two methods are used to remove above said difficulty.

First method : Substitute a small positive number `ε' in place of a zero occurred as a first element in a row. Complete the array with this number `ε'. Then examine the sign change by taking $\lim_{\varepsilon \to 0}$. Consider above example.

s^5	1	3	2
s^4	2	6	1
s^3	ε	1.5	0
s^2	$\frac{6\varepsilon - 3}{\varepsilon}$	1	0
s^1	$\dfrac{\frac{1.5(6\varepsilon - 3)}{\varepsilon} - \varepsilon}{\frac{(6\varepsilon - 3)}{\varepsilon}}$	0	
s^0	1		

To examine sign change,

$$\lim_{\varepsilon \to 0}\left(\frac{6\varepsilon - 3}{\varepsilon}\right) = 6 - \lim_{\varepsilon \to 0}\frac{3}{\varepsilon}$$

$$= 6 - \infty$$

$$= -\infty \text{ sign is negative.}$$

$$\lim_{\varepsilon \to 0}\frac{1.5(6\varepsilon - 3) - \varepsilon^2}{6\varepsilon - 3} = \lim_{\varepsilon \to 0}\frac{9\varepsilon - 4.5 - \varepsilon^2}{6\varepsilon - 3}$$

$$= \frac{0 - 4.5 - 0}{0 - 3}$$

$= +1.5$ sign is positive.

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Routh's array is,

s^5	1	3	2
s^4	2	6	1
s^3	$+\varepsilon$ ↓	1.5	0
s^2	$-\infty$ ↓	1	0
s^1	+ 1.5	0	0
s^0	1	0	0

As there are two sign changes, **system is unstable.**

Example

Find range of values of `K' so that system with the following characteristic equation will be stable. $F(s) = s(s^2 + s + 1)(s + 4) + K = 0$

Sol. :

$$F(s) = s[s^3 + 5s^2 + 5s + 4] + K = 0$$

$$= s^4 + 5s^3 + 5s^2 + 4s + K = 0$$

s^4	1	5	K
s^3	5	4	0
s^2	4.2	K	0
s^1	$\frac{16.8 - 5K}{4.2}$		
s^0	K		

For system to be stable there should not be sign change in the first column.

$\therefore \quad K > 0 \quad$ from s^0

and $\quad 16.8 - 5K > 0 \quad$ from s^1

$\therefore \quad 16.8 > 5K \quad \therefore \quad 3.36 > K \quad \therefore \; K < 3.36$

$\therefore$ Range of `K' is $0 < K < 3.36$

Example : $s^6 + 4s^5 + 3s^4 - 16s^2 - 64s - 48 = 0$ *Find the number of roots of this equation with positive real part, zero real part and negative real part.*

Solution :

s^6	1	3	− 16	− 48
s^5	4	0	− 64	0
s^4	3	0	− 48	0
s^3	0	0	0	

$$A(s) = 3s^4 - 48 = 0 \qquad \frac{dA}{ds} = 12s^3$$

Routh's array is,

s^6	1	3	− 16	− 48
s^5	4	0	− 64	0
s^4	3	0	− 48	0
s^3	12	0	0	0
s^2	[ε] 0	− 48	0	0
s^1	$\frac{576}{\varepsilon}$	0	0	
s^0	− 48			

$$\lim_{\varepsilon \to 0} \frac{576}{\varepsilon} = +\infty$$

∴ One sign change and **system is unstable**. Thus there is one root in R.H.S. of s-plane i.e. with positive real part. Now solve A(s) = 0 for the dominant roots.

$A(s) = 3s^4 - 48 = 0$

Put $s^2 = y$

∴ $3y^2 = 48$ ∴ $y^2 = 16,$ ∴ $y = \pm\sqrt{16} = \pm 4$

∴ $s^2 = +4$ $s^2 = -4$

$s = \pm 2$ $s = \pm 2j$

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So $s = \pm 2j$ are the two roots on imaginary axis i.e. with zero real part. Root in R.H.S. indicated by a sign change is $s = +2$ as obtained by solving $A(s) = 0$. Total there are 6 roots as $n = 6$.

Roots with positive real part = 1

Roots with zero real part = 2

Roots with negative real part = 6 – 2 – 1 = 3

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