STEADY-STATE ERRORS IN UNITY-FEEDBACK CONTROL SYSTEMS

Any physical control system inherently suffers steady-state error in response to certain types of inputs. A system may have no steady-state error to a step input, but the same system may exhibit nonzero steady-state error to a ramp input. (The only way we may be able to eliminate this error is to modify the system structure.) Whether a given system will exhibit steady-state error for a given type of input depends on the type of open-loop transfer function of the system.

Classification of Control Systems.

Control systems may be classified according to their ability to follow step inputs, ramp inputs, parabolic inputs. The magnitudes of the steady-state errors due to these individual inputs are indicative of the goodness of the system.

Consider the unity-feedback control system with the following open-loop transfer function G(s):

$$G(s) = \frac{K(T_a s + 1)(T_b s + 1)\cdots(T_m s + 1)}{s^N(T_1 s + 1)(T_2 s + 1)\cdots(T_p s + 1)}$$

A system is called:

type 0, if N=0,

type 1, if N=1,

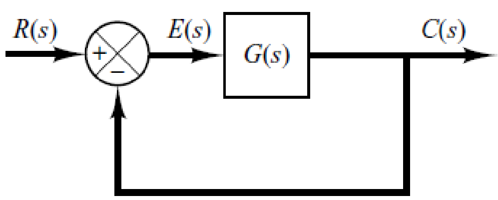
type 2, if N=2,

type 3, if N=3,

the open-loop gain K is directly related to the steady-state error.

Steady-State Errors. Consider the system shown in Figure below. The closed-loop transfer function is

$$\frac{C(s)}{R(s)} = \frac{G(s)}{1 + G(s)}$$



The transfer function between the error signal e(t) and the input signal r(t) is

$$\frac{E(s)}{R(s)} = \frac{R(s) - C(s)}{R(s)} = 1 - \frac{C(s)}{R(s)} = 1 - \frac{G(s)}{1 + G(s)} = \frac{1 + G(s) - G(s)}{1 + G(s)}$$

$$\frac{E(s)}{R(s)} = \frac{1}{1+G(s)}$$

$$E(s) = \frac{1}{1 + G(s)} R(s)$$

The final-value theorem provides a convenient way to find the steady-state performance of a stable system.

$$e_{ss} = \lim_{t \to \infty} e(t) = \lim_{s \to 0} sE(s) = \lim_{s \to 0} \frac{sR(s)}{1 + G(s)}$$

Static Position Error Constant *Kp*. The steady-state error of the system for a unit-step input is

$$e_{ss} = \lim_{s \to 0} \frac{s}{1 + G(s)} \frac{1}{s} = \frac{1}{1 + G(0)}$$

The static position error constant Kp is defined by

$$K_p = \lim_{s \to 0} G(s) = G(0)$$

Thus, the steady-state error in terms of the static position error constant Kp is given by

$$e_{ss} = \frac{1}{1 + K_p}$$

For a type 0 system,

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1) \cdots}{(T_1 s + 1)(T_2 s + 1) \cdots} = K$$

$$e_{ss} = \frac{1}{1 + K},$$

For a type 1 or higher system,

$$K_p = \lim_{s \to 0} \frac{K(T_a s + 1)(T_b s + 1) \cdots}{s^N (T_1 s + 1)(T_2 s + 1) \cdots} = \infty, \qquad \text{for } N \geq 1$$

$$e_{ss} = 0,$$

Static Velocity Error Constant *Kv*. The steady-state error of the system with a unit-ramp input is given by

$$e_{ss} = \lim_{s\to 0} \frac{s}{1 + G(s)} \frac{1}{s^2} = \lim_{s\to 0} \frac{1}{sG(s)}$$

The static velocity error constant *K*v is defined by

$$K_v = \lim_{s\to 0} sG(s)$$

Thus, the steady-state error in terms of the static velocity error constant *K*v is given by

$$\boldsymbol{e_{ss} = \frac{1}{K_v}}$$

For a type 0 system,

$$K_v = \lim_{s\to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

$$e_{ss} = \frac{1}{K_v} = \infty,$$

For a type 1 system,

$$K_v = \lim_{s\to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = K$$

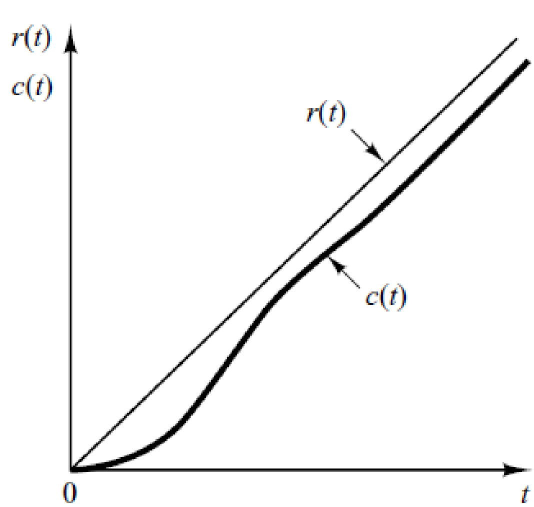
$$e_{ss} = \frac{1}{K_v} = \frac{1}{K},$$

For a type 2 or higher system,

$$K_v = \lim_{s\to 0} \frac{sK(T_a s + 1)(T_b s + 1)\cdots}{s^N(T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \quad \text{for } N \geq 2$$

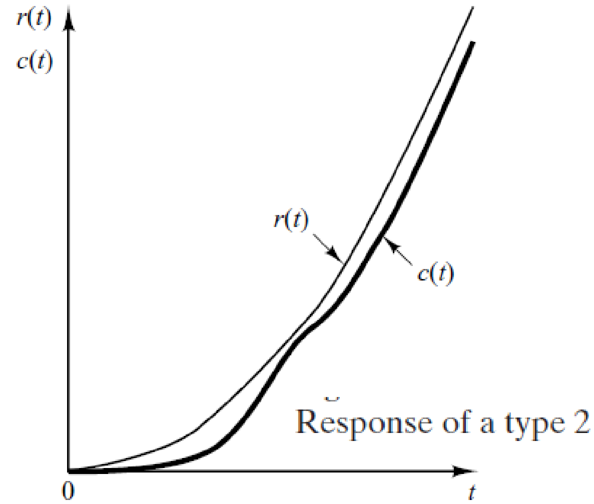
$$e_{ss} = \frac{1}{K_v} = 0,$$

Response of a type 1



Static Acceleration Error Constant *Ka*. The steady-state error of the system with a unit-parabolic input (acceleration input), which is defined by

$$r(t) = \frac{t^2}{2}, \quad \text{for } t \geq 0$$
$$= 0, \quad \text{for } t < 0$$



$$e_{ss} = \lim_{s\to 0} \frac{s}{1 + G(s)} \frac{1}{s^3} = \frac{1}{\lim_{s\to 0} s^2 G(s)}$$

The static acceleration error constant *Ka* is defined by the equation

$$K_a = \lim_{s\to 0} s^2 G(s)$$

$$e_{ss} = \frac{1}{K_a}$$

For a type 0 system,

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

$$e_{ss} = \infty,$$

For a type 1 system

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s(T_1 s + 1)(T_2 s + 1)\cdots} = 0$$

$$e_{ss} = \frac{1}{K},$$

For a type 3 or higher system,

$$K_a = \lim_{s\to 0} \frac{s^2 K(T_a s + 1)(T_b s + 1)\cdots}{s^N (T_1 s + 1)(T_2 s + 1)\cdots} = \infty, \quad \text{for } N \geq 3$$

$$e_{ss} = 0,$$

The steady-state errors for type 0, type 1, and type 2 systems when they are subjected to various inputs.

Steady-State Error in Terms of Gain K

	Step Input $r(t) = 1$	Ramp Input $r(t) = t$	Acceleration Input $r(t) = \frac{1}{2}t^2$
Type 0 system	$\frac{1}{1+K}$	∞	∞
Type 1 system	0	$\frac{1}{K}$	∞
Type 2 system	0	0	$\frac{1}{K}$

Ex: The forward T.F of a unity F.B. type 1, second order system has a pole at −2. The nature of gain K is so adjusted that damping ratio is 0.4. The above system is subjected to input $r(t) = 1 + 4t$. Find $e_{s.s}$ $G(s) = \frac{K}{s(s+2)}$

Solution

$$G(s) = \frac{K}{s(s+2)} , \quad \frac{C(s)}{R(s)} = \frac{K}{s^2 + 2s + K}$$

$$2\zeta\omega_n = 2 \ ; \ \omega_n = \sqrt{K} \ ; \ \zeta = 0.4 \ , \ 0.8\sqrt{K} = 2 \ , \ \therefore K = 6.25$$

$$G(s) = \frac{6.25}{s(s+2)} , \quad K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{6.25}{s(s+2)} = \infty$$

$$K_v = \lim_{s \to 0} s\,G(s) = 6.25/2 = 3.125$$

$$\therefore e_{ss} = \frac{R}{1+K_p} + \frac{R}{K_v} = \frac{1}{1+\infty} + \frac{4}{3.12} = \boxed{1.28}$$

Ex The open loop T.F of a unity F.B system is given by

$$G(s) = \frac{100}{s(s+10)}$$

Find the static error constant and the steady state error of the system when subjected to an input given by the polynomial

$$r(t) = P_0 + P_1 t + \frac{P_2}{2} t^2$$

Solution

$$G(s) = \frac{100}{s(s+10)}$$

$$K_p = \lim_{s \to 0} G(s) = \lim_{s \to 0} \frac{100}{s(s+10)} = \infty$$

$$K_v = \lim_{s \to 0} s\,G(s) = \lim_{s \to 0} \frac{100}{s+10} = 10$$

$$K_a = \lim_{s \to 0} s^2 G(s) = \lim_{s \to 0} \frac{100}{s+10} = 0$$

$$r(t) = P_0 + P_1 t + \frac{P_2}{2} t^2$$

$$e_{ss} = \frac{R_1}{1+K_p} + \frac{R_2}{K_v} + \frac{R_3}{K_a}$$

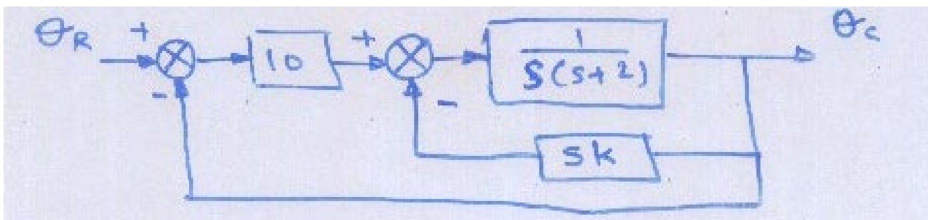
$$= \frac{P_0}{1+\infty} + \frac{P_1}{10} + \frac{P_2}{2 \times 0} = \infty$$

Example

A system shown employs derivative of F.B in addition to unity F.B

a) Find the value of constant k so that the damping ratio of the sys. is 0.7. Then find the value of the steady state error with unit ramp input.

b) For $k=0$ Calculate ξ, ω_n & e_{ss} with unit ramp input



$$\frac{\theta_C}{\theta_R}(s) = \frac{10}{s^2 + s(2+K) + 10}$$

$\therefore \omega_n = \sqrt{10} = 3.16$ rad sec

$2+k = 2\xi\omega_n = 2 \times 0.7 \times 3.16 \Rightarrow \therefore \boxed{K = 2.424}$

$e_{ss} = \frac{1}{K_v}$, $\because K_v = \lim_{s \to 0} s\,G(s)\,H(s)$, $\because H(s) = 1$

$$G(s) = \frac{10}{s^2+2s} \Big/ \left(1 + \frac{sk}{s^2+2s}\right) = \frac{10}{s^2 + s(2+k)}$$

$G(s) = \frac{10}{s(s+2+2.42)}$, $\therefore K_v = \lim_{s \to 0} \frac{10}{s+4.42}$

$K_v = 2.26$, $\therefore e_{s.s} = \frac{1}{2.26} = 0.44$ rad

b) for $k=0$

$G(s) = \frac{10}{s(s+2)}$, $\therefore \frac{\theta_c}{\theta_R}(s) = \frac{10}{s^2+2s+10}$

$\omega_n = \sqrt{10} = 3.16$ rad/sec , $2\xi\omega_n = 2 \Rightarrow \xi = 0.316$

$K_v = \lim_{s \to 0} s\,G(s) = \frac{10}{2} = 5$, $e_{ss} = 0.2$ radian.