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Homogenous First - Order Differential Equation

An equation of the form &

$$\frac{dy}{dx} = F\left(\frac{y}{x}\right) \quad \text{--- (1)}$$

is called a homogenous equation. In this equation, the variables cannot be separated but can be transformed by a change of variable into an equation where the variables can be separated.

To transform a homogenous equation such as equation (1) into an equation whose variables may be separated we introduce the new independent variable &

$$y = u \cdot x \Rightarrow dy = u \cdot dx + x \cdot du$$

عند إجراء هذا التعويض فإن المعادلة التفاضلية دائماً تتحول إلى دالة قابلة للفصل وبذلك يسهل حلها. وبعد الحصول على الحل نعوّض بـ u بـ y .

$$u = \frac{y}{x}.$$

كأن المعادلة التفاضلية متجانسة إذا كانت الدالة $M(x, y)$ والدالة $N(x, y)$ متجانسة أيضاً وبمعنى الدرجة ويتم التحقق من ذلك عن خلال تعويض x بـ λx و y بـ λy فتتحقق ما يلي &

$$M(\lambda x, \lambda y) = \lambda \cdot M(x, y)$$

$$N(\lambda x, \lambda y) = \lambda \cdot N(x, y)$$

This part is homogenous.



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Ex(1) g- Find the general solution for $(x^2 + y^2) dx + xy dy = 0$

$$(x^2 + y^2) dx = -xy dy$$

$$M(x, y) = x^2 + y^2$$

$$M(\lambda x, \lambda y) = \lambda^2 x^2 + \lambda^2 y^2 = \lambda^2 (x^2 + y^2) = \lambda^2 M(x, y). \quad \therefore \text{homogenous.}$$

$$N(x, y) = -xy$$

$$N(\lambda x, \lambda y) = -(\lambda x \cdot \lambda y) = -\lambda^2 (xy) = \lambda^2 N(x, y). \quad \therefore \text{homogenous.}$$

To solve this equation, assume

$$y = u \cdot x \Rightarrow dy = u dx + x du$$

$$(x^2 + u^2 x^2) dx = -x(u \cdot x) (u dx + x du)$$

$$x^2 dx + u^2 x^2 dx = -(x^2 u^2 dx + x^3 u du)$$

~~$$x^2 dx + u^2 x^2 dx = -x^2 u^2 dx - x^3 u du$$~~

~~$$x^2 dx + u^2 x^2 dx = -x^2 u^2 dx - x^3 u du$$~~

$$(x^2 + 2x^2 u^2) dx = -x^3 u du$$

$$(1 + 2u^2) x^2 dx = -x^3 u du$$

$$\frac{dx}{x} = \frac{-u}{1 + 2u^2} du \quad \left\{ \text{by integration} \right\}$$

$$\ln x = -\frac{1}{4} \ln(1 + 2u^2) + C \quad] * 4$$

$$4 \ln x + \ln(1 + 2u^2) = C_2$$

$$C_2 = 4C$$



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Ex 2. Find the general solution for $y^2 dx + x^2 dy = 2xy dy$?

$$y^2 dx = (2xy - x^2) dy \quad (*)$$

$$M(x, y) = y^2$$

$$M(\lambda x, \lambda y) = \lambda^2 y^2 = \lambda^2 M(x, y) \Rightarrow \text{homogenous.}$$

$$N(x, y) = 2xy - x^2$$

$$N(\lambda x, \lambda y) = 2\lambda x \cdot \lambda y - \lambda^2 x^2 = \lambda^2 (2xy - x^2) = \lambda^2 N(x, y) \Rightarrow \text{homogenous}$$

from eq. (*)

$\Rightarrow$ The total equation is homogenous.

$$\frac{dy}{dx} = \frac{y^2}{2xy - x^2} \Rightarrow \text{To solve this equation, assume:}$$

$$y = u \cdot x \quad \text{and} \quad dy = u \cdot dx + x \cdot du$$

$$u^2 x^2 dx + x^2 (u \cdot dx + x \cdot du) = 2x (u \cdot x) du$$

$$u^2 x^2 dx + x^2 u dx = 2u x^2 du - x^3 du$$

$$(u^2 + u) x^2 dx = x^3 (2u - 1) du$$

$$\frac{x^2}{x^3} dx = \left(\frac{2u-1}{u+u^2} \right) du = \frac{1}{x} dx = \left(\frac{2u-1}{u+u^2} \right) du$$

$$\ln x = \ln(u + u^2) + C_1 \Rightarrow \ln x - \ln(u + u^2) = C_1$$

$$\ln \frac{x}{u + u^2} = C_1 \quad \left. \begin{array}{l} \text{نأخذ } e \text{ للطرفين} \end{array} \right\}$$

$$\frac{x}{u + u^2} = e^C \Rightarrow \frac{x}{u + u^2} = C \quad [C = e^C]$$



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but $u = \frac{y}{x}$ lead to

$$\frac{x}{\frac{y}{x} + \frac{y^2}{x^2}} = c \Rightarrow \boxed{x^3 - y(x-y)c = 0} \quad G.E$$

Ex(3) 8- Find the general solution for
 $x^2 dy + (y^2 - xy) dx = 0$
 $x^2 dy = -(y^2 - xy) dx$

$$\begin{aligned} M(x,y) &= -(y^2 - xy) \\ M(\lambda x, \lambda y) &= -(\lambda^2 y^2 - \lambda x \lambda y) \\ &= -\lambda^2 (y^2 - xy) \\ &= \lambda^2 M(x,y) \quad \therefore \text{homogenous} \end{aligned}$$

$$\begin{aligned} N(x,y) &= x^2 \\ N(\lambda x, \lambda y) &= \lambda^2 x^2 \\ &= \lambda^2 N(x,y) \quad \therefore \text{homogenous} \end{aligned}$$

$\therefore$ The total equation is homogenous

$$\frac{dy}{dx} = - \frac{(y^2 - xy)}{x^2} \Rightarrow \text{To solve this equation, assume}$$

$$y = u \cdot x \quad \text{and} \quad dy = u \cdot dx + x \cdot du$$

$$\begin{aligned} x^2 (u \cdot dx + x \cdot du) &= - (u^2 x^2 - x^2 u) dx \\ \cancel{x^2 u dx} + x^3 du + u^2 x^2 dx - \cancel{x^2 u dx} &= 0 \\ x^3 du + u^2 x^2 dx &= 0 \\ u^2 x^2 dx &= -x^3 du \\ \frac{dx}{x} &= -\frac{1}{u^2} du \Rightarrow \int \frac{1}{x} dx = -\int \frac{1}{u^2} du \end{aligned}$$



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$$\ln x + \frac{1}{u} = c \Rightarrow \boxed{\ln x + \frac{x}{y} = c}$$

Ex(4):- Find the general solution for
 $(x+y) dy + (x-y) dx = 0$

$$M(x,y) = (x-y)$$

$$M(\lambda x, \lambda y) = (\lambda x - \lambda y) \Rightarrow \lambda (x-y)$$

$$\therefore M(\lambda x, \lambda y) = \lambda M(x,y)$$

$\therefore$ homogeneous.

$$N(x,y) = (x+y)$$

$$N(\lambda x, \lambda y) = (\lambda x + \lambda y) = \lambda (x+y) = \lambda N(x,y)$$

$\therefore$ homogeneous.

$\therefore$ The total equation is homogeneous.

$$\frac{dy}{dx} = -\frac{(x-y)}{(x+y)} \Rightarrow \text{To solve this equation, assume}$$

$$y = u \cdot x \Rightarrow dy = u \cdot dx + x \cdot du$$

$$(x + ux) (u \cdot dx + x \cdot du) + (x - ux) dx = 0$$

$$xu dx + x^2 du + u^2 x dx + ux^2 du + x dx - ux dx = 0$$

$$(xu + u^2 x + ux + x) dx + (x^2 + ux^2) du = 0$$

$$(u^2 x + x) dx + (1 + u) x^2 du = 0$$

$$(u^2 + 1) dx + (1 + u) x du = 0$$

$$\frac{1}{x} dx + \frac{1+u}{1+u^2} du = 0$$



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$$\frac{1}{x} dx + \frac{1}{1+u^2} du + \frac{u}{1+u^2} du = 0 \quad \left. \vphantom{\frac{1}{x} dx} \right\} \text{by integra)}$$

$$\ln x + \tan^{-1} u + \frac{1}{2} \ln(1+u^2) + C = 0$$

$$\boxed{\ln x + \tan^{-1} \frac{y}{x} + \frac{1}{2} \ln\left(1 + \frac{y^2}{x^2}\right) + C = 0}$$

How & Find the general solution for

$$1- (xy + y^2) dx = (x^2 + xy + y^2) dy.$$

Ans.

$$\ln x + \frac{1}{2} \frac{y^2}{x^2} - \frac{x}{y} + \ln \frac{y}{x} + C = 0$$

$$2- (3y^3 - x^3) dx = 3xy^2 dy, \quad y(1) = 2$$

Ans.

$$\ln x + \frac{1}{3} \frac{y^3}{x^3} = \frac{8}{3}.$$

$$3- \frac{dy}{dx} = \frac{y+2}{x+y+1}$$

Ans.

$$\ln(x+1) + \frac{x-1}{y+2} + \ln \frac{y+2}{x-1} = C$$