Ordinary Differential Equations

– First order Differential Equations (F.O.D.E):
is defined by an equation $\frac{dy}{dx} = f(x,y)$ of two variables x and y.

Types of F.O.D.E:

1– Variable Separation
2– Homogenous eq.
3– Exact eq.
4– linear eq.
5– Bernoulli's eq.

Applications of F.O.D.E.

– Newton's law of cooling
– Growth and decay
– Electrical circuts
– Falling body problems
– Dilution problems.

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1. Variable Separation

This form is:

$$\int f(x)\,dx + \int g(y)\,dy = C$$

Ex.1/ solve; $(x^2+1)\dfrac{dy}{dx} = xy$.

Sol.

$$(x^2+1)\,dy = xy\,dx \qquad (\div\,(x^2+1)y)$$

$$\frac{(x^2+1)}{(x^2+1)y}\,dy = \frac{xy}{(x^2+1)y}\,dx$$

$$\Rightarrow \int \frac{1}{y}\,dy = \int \frac{x}{(x^2+1)}\,dx$$

$$\ln y = \frac{1}{2}\ln(x^2+1) + C$$

$$\ln|y| = \ln(x^2+1)^{\frac{1}{2}} + C \qquad (\text{take exp.})$$

$$y = \sqrt{x^2+1} * A \qquad (A = e^{c})$$

$$\therefore y = A\sqrt{x^2+1}$$

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Ex.2/ Find the Solution of $e^{x}\frac{dy}{dx}=4$, that subjected to $y(0)=3$.

Solution:

$$e^{x}\,dy=4\,dx$$

$$dy=\frac{4}{e^{x}}\,dx$$

$$\int dy=\int 4e^{-x}\,dx$$

$$\therefore y=-4e^{-x}+c$$

B.C.

$$3=-4e^{0}+c \Rightarrow c=7$$

$$\therefore y=-4e^{-x}+7$$

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2. Homogenous eq.

It can be solved by introducting a new dependent variable.

$\rightarrow y = ux \quad \rightarrow \quad dy = u\,dx + x\,du$

Ex. 1/ solve; $2x\,dy = (x+y)\,dx$

Solution:

$y = ux$ and $dy = u\,dx + x\,du$] sub. in D.E

$$2x(u\,dx + x\,du) = (x + ux)\,dx$$

$$2x(u\,dx + x\,du) = x(1+u)\,dx \quad] \div x$$

$$2u\,dx + 2x\,du = dx + u\,dx$$

$$u\,dx + 2x\,du = dx$$

$$2x\,du = dx - u\,dx$$

$$2x\,du = (1-u)\,dx \quad] \div 2x(1-u)$$

$$\int \frac{1}{(1-u)}\,du = \int \frac{1}{2x}\,dx \Rightarrow -\ln(1-u) = \frac{1}{2}\ln x + C$$

$$\ln(1-u)^{-1} = \ln x^{\frac{1}{2}} + \ln C$$

$$\ln\left(\frac{1}{1-u}\right) = \ln|\sqrt{x}\cdot c| \quad] \text{ take exp.}$$

$$\Rightarrow \frac{1}{1-u} = c\sqrt{x}$$

$y = ux$
$\rightarrow u = \frac{y}{x}$

$$\therefore \frac{1}{1-\frac{y}{x}} = c\sqrt{x}$$

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Ex. 2/ Solve: $\frac{dy}{dx} = \frac{xy}{x^2 - y^2}$

Solution: $xy\,dx = (x^2 - y^2)\,dy$

$y = ux$ and $dy = x\,du + u\,dx$] sub in D.E.

$$u x^2 dx = (x^2 - u^2 x^2)(x\,du + u\,dx)$$

$$u x^2 dx = x^2(1 - u^2)(x\,du + u\,dx) \quad (\div x^2)$$

$$\cancel{u\,dx} = x\,du + \cancel{u\,dx} - x u^2 du - u^3 dx$$

$$u^3 dx = x\,du - x u^2 du$$

$$u^3 dx = x(1 - u^2)\,du \quad (\div u^3 x)$$

$$\rightarrow \frac{1}{x}dx = \frac{1 - u^2}{u^3}du$$

$$\int \frac{1}{x}dx = \int \frac{1}{u^3}du - \int \frac{1}{u}du$$

$$\ln|x| = \frac{-1}{2u^2} - \ln|u| + c$$

$$\ln|x| = \frac{-1}{2\frac{y^2}{x^2}} - \ln\left|\frac{y}{x}\right| + c$$

$$\cancel{\ln|x|} = \frac{-x^2}{2y^2} - \ln|y| + \cancel{\ln|x|} + c$$

$$\ln|y| = \frac{-x^2}{2y^2} + c \longrightarrow y = e^{\frac{-x^2}{2y^2}} * e^c \quad \{A = e^c\}$$

$$\therefore \quad y = A\,e^{\frac{-x^2}{2y^2}}$$

③ Exact eq.

Standard form: $M(x,y)\,dx + N(x,y)\,dy = 0$ --- ①

Eq-1 is Exact if $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

Ex.1 / Solve $(2xy^2-4)\,dx + (2x^2y+3)\,dy = 0$

Sol/

$$\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$$

$$\frac{\partial M}{\partial y} = 4xy \quad , \quad \frac{\partial N}{\partial x} = 4xy$$

$\therefore$ Exact $\quad \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$$\boxed{\int^{x} M(x,y)\,dx + \int^{y} N(x,y)\,dy - \int^{y}\left[\frac{\partial}{\partial y}\int^{x} M(x,y)\,dx\right]dy}$$

① $\int^{x} M(x,y)\,dx = \int^{x}(2xy^2-4)\,dx$
$= x^2y^2 - 4x$

② $\int^{y} N(x,y)\,dy = \int^{y}(2x^2y+3)\,dy =$
$= x^2y^2 + 3y$

③ $\frac{\partial}{\partial y}\left[x^2y^2 \cancel{-4x}\right] = 2yx^2$

④ $\int^{y} 2yx^2\,dy = y^2x^2$

$= x^2y^2 - 4x + \cancel{x^2y^2} + 3y - y^2x^2$

$= x^2y^2 - 4x + 3y = f(x,y)$

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Ex.2 // Solve : $(4x^3y^3 - 2xy)dx + (3x^4y^2 - x^2)dy = 0$

Sol

$$M = 4x^3y^3 - 2xy$$
$$\frac{\partial M}{\partial y} = 12y^2x^3 - 2x$$
$$N = 3x^4y^2 - x^2$$
$$\frac{\partial N}{\partial x} = 12x^3y^2 - 2x$$

$\therefore$ Exact equation

$$= \boxed{\int^x M(x,y)\,dx + \int^y N(x,y)\,dy - \int^x \left[\frac{\partial}{\partial x}\int^y N(x,y)\,dy\right]dx}$$

① $\int^x M(x,y)\,dx = \int (4x^3y^3 - 2xy)\,dx$
$$= x^4y^3 - x^2y$$

② $\int^y N(x,y)\,dy = \int (3x^4y^2 - x^2)\,dy$
$$= x^4y^3 - x^2y$$

③ $\frac{\partial}{\partial x}[x^4y^3 - x^2y] = 4x^3y^3 - 2xy$

④ $\int [4x^3y^3 - 2xy]\,dx$
$$= x^4y^3 - x^2y$$

$$= x^4y^3 - x^2y + \cancel{x^4y^3} - \cancel{x^2y} - \cancel{x^4y^3} + \cancel{x^2y}$$

$$f(x,y) = x^4y^3 - x^2y$$

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4- Linear eq.

General form: $\frac{dy}{dx} + P(x)\,y = Q(x)$; to solve it:-

1- Determine Integration Factor $I.F = e^{\int P(x)\,dx}$

2- Integrate and Solve $y(x) = \frac{1}{I.F}\int I.F \times Q(x)\,dx + \frac{c}{I.F}$

EX.1/ Solve; $\frac{dy}{dx} + \frac{3y}{x} = x^2$ at $y(1) = \frac{1}{6}$

Sol:

$P(x) = \frac{3}{x}$, $Q(x) = x^2$

$I.f = e^{\int \frac{3}{x}dx} \rightarrow I.f = e^{3\int \frac{1}{x}dx} \rightarrow I.f = e^{3\ln|x|}$

$\therefore If = x^3$

$y(x) = \frac{1}{x^3}\int x^3 \times x^2\,dx + \frac{c}{x^3}$

$= \frac{1}{\cancel{x^3}}\left[\frac{x^{\cancel{6}\,3}}{6}\right] + \frac{c}{x^3}$

$y(x) = \frac{x^3}{6} + \frac{c}{x^3}$

$\frac{1}{6} = \frac{1^3}{6} + \frac{c}{1^3} \Longrightarrow c = 0$

$\therefore y = \frac{x^3}{6}$

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Ex-2/ Solve the D.E. $x\frac{dy}{dx} - y = x^2\cos x$ at $y(\frac{\pi}{2}) = \pi$

Solution:

$$x\frac{dy}{dx} - y = x^2\cos x \xrightarrow{\div x} \frac{dy}{dx} - \frac{y}{x} = x\cos x$$

$$I.f = e^{\int \frac{-1}{x}dx} \rightarrow I.f = e^{-\int \frac{1}{x}dx} \rightarrow I.f = e^{-\ln(x)}$$

$$\therefore I.f = \frac{1}{x}$$

$$y = x\left[\int \frac{1}{x} \cdot x\cos x\, dx\right] + xc$$

$$y = x\sin x + xc$$

B.C.

$$\pi = \frac{\pi}{2}\left(\sin\frac{\pi}{2} + c\right) \quad (*2)$$

$$2\cancel{\pi} = \cancel{\pi}\left(\sin\frac{\pi}{2} + c\right)$$

$$2 = \sin\frac{\pi}{2} + c$$

$$2 = 1 + c \longrightarrow \boxed{c = 1}$$

$$\therefore y = x\sin x + x$$

5- Bernoulli's equation

Standard form: $\frac{dy}{dx} + P(x)\, y = Q(x)\, y^n$

to solve it, we set $z = y^{1-n}$

Ex. Solve $\frac{dy}{dx} + \frac{1}{x} y = x y^2$

Sol:

$z = y^{1-2} \rightarrow z = y^{-1} \rightarrow z = \frac{1}{y}$ and $y = \frac{1}{z}$

$dy = \frac{-1}{z^2}\, dz$ (Sub in D.E)

$\frac{-1}{z^2}\frac{dz}{dx} + \frac{1}{xz} = x\frac{1}{z^2}$ $(* -z^2)$

$\frac{dz}{dx} - \frac{z}{x} = -x$ (linear form)

$I.f = e^{\int -\frac{1}{x} dx} \rightarrow e^{-\ln x} \rightarrow I.f = \frac{1}{x}$

$z(x) = x \int \frac{1}{x} * (-x)\, dx + cx$

$z(x) = -x^2 + cx$

$\frac{1}{y} = -x^2 + cx$

$\therefore y = \frac{1}{-x^2 + cx}$

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Example (4) Solve $\dfrac{\partial^3 Z}{\partial x^2 \partial y} = 18 x y^2 + \sin(2x-y)$

Sol: Integrate w.r. to x

$$\frac{\partial^2 Z}{\partial x \partial y} = 18 \frac{x^2}{2} y^2 - \frac{\cos(2x-y)}{2} + f(y)$$

Integrate w.r. to x

$$\frac{\partial Z}{\partial y} = 9 \frac{x^3}{3} y^2 - \frac{1}{2} \frac{\sin(2x-y)}{2} + F(y)$$

Integrate w.r. to y

$$Z = 3x^3 \frac{y^3}{3} - \frac{1}{4} \frac{[-\cos(2x-y)]}{-1} + F(y) + g(x)$$

$$\therefore Z = x^3 y^3 - \frac{1}{4} \cos(2x-y) + F(y) + g(x)$$

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