The z-Transform

1.1. Introduction

The z-transform is a useful tool in the analysis of discrete-time signals and systems and is the discrete-time counterpart of the Laplace transforms for continuous-time signals and systems. The z-transform may be used to solve constant coefficient difference equations, evaluate the response of a linear time-invariant system to a given input, and design linear filters.

1.2. Definition of the z-Transform

The discrete-time Fourier transform (DTFT) of a sequence x(n) is equal to the sum:

$$X\left(e^{jw}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-jnw}$$

However. for this series to converge, it is necessary that the signal be summable. Unfortunately, many of the signals that we would like to consider are not summable and, therefore, do not have a DTFT. Some examples include:

$$x(n) = u(n),\ x(n) = (0.5)^n u(-n)$$

The z-transform is a generalization of the DTFT that allows one to deal with such sequences, the z-transform of a discrete-time signal x(n) is defined by:

$$X(z) = \sum_{n=0}^{\infty} x(n)z^{-n}$$

Where,

$z = re^{jw}$ is a complex variable. The values of z for which the sum converges define a region in the z-plane referred to as the region of convergence (ROC)

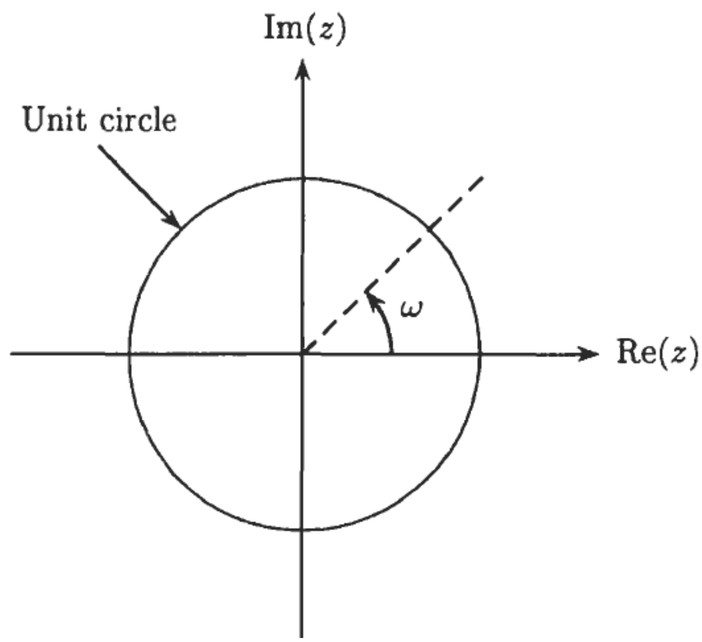

Figure 1.1. The unit circle in the complex z-plane

Because the z-transform is a function of a complex variable, it is convenient to describe it using the complex z-plane. With

$$z = Re(z) + jIm(z) = re^{jw}$$

The axes of the z-plane are the real and imaginary parts of ***z*** as illustrated in Fig.1.1, and the contour corresponding to $|z| = 1$ is a circle of unit radius referred to as the **unit circle**. The z-transform evaluated on the unit circle corresponds to the DTFT,

$$X\left(e^{jw}\right) = X(z)]_{z=e^{jw}}$$

More specifically, evaluating X(z) at points around the unit circle, beginning at z=1(w=0), through z=j ($w=\pi/2$), to z=-1(w= π), we obtain the values of $X(e^{jw})$ for $0\leq w \leq \pi$. Note that for the DTFT of a signal to exist, the unit circle must be within the region of convergence of X(z).

Example1: Given the sequence

Find the z-transform of *x(n)*.

$$x(n) = u(n),$$

Solution:

From the definition, the z-transform is given by

$$X(z) = \sum_{n=0}^{\infty} u(n) z^{-n} = \sum_{n=0}^{\infty} (z^{-1})^n = 1 + (z^{-1}) + (z^{-1})^2 + \dots$$

This is an infinite geometric series that converges to

$$X(z) = \frac{z}{z-1}$$

The region of convergence for all values of z is given as $|z| > 1$

zahraa.eisa@uomus.edu.iq

Example2: Considering the exponential sequence

$$x(n) = a^n u(n),$$

Find the z-transform of the sequence *x(n)*.

Solution:

$$X(z) = \sum_{n=0}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = 1 + \left(az^{-1}\right) + \left(az^{-1}\right)^2 + \ldots.$$

Since this is a geometric series which will converge for $|az^{-1}| < 1$ it is further expressed as

Table 1.1. Common z-Transform Pairs

Sequence	z-Transform	Region of Convergence
$\delta(n)$	1	all z
$\alpha^n u(n)$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-\alpha^n u(-n-1)$	$\dfrac{1}{1-\alpha z^{-1}}$	$\lvert z\rvert < \lvert\alpha\rvert$
$n\alpha^n u(n)$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert > \lvert\alpha\rvert$
$-n\alpha^n u(-n-1)$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\lvert z\rvert < \lvert\alpha\rvert$
$\cos(n\omega_0)u(n)$	$\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$
$\sin(n\omega_0)u(n)$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\lvert z\rvert > 1$

zahraa.eisa@uomus.edu.iq

1.3. Properties of z-Transform

1. **Linearity:** the z-transform is a linear operator. Therefore, if x(n) has a z-transform X(z) with a region of convergence R_x, and if y(n) has a z-transform Y (z) with a region of convergence R_y,

$$w(n) = ax(n) + by(n) \stackrel{Z}{\longleftrightarrow} W(z) = aX(z) + bY(z)$$

and the ROC of w(n) will include the intersection of R_x, and R_y, that is,

$$R_w \text{ contains } R_x \cap R_y$$

Example3: Find the z-transform of the sequence defined by

$$x(n) = u(n) - (0.5)^n u(n).$$

Solution:
Applying the linearity of the z-transform previously discussed, we have

$$x(n) = u(n) - (0.5)^n u(n).$$

From the Table1

$$Z(u(n)) = \frac{z}{z-1}$$

$$\text{and } Z(0.5^n u(n)) = \frac{z}{z-0.5}.$$

Substituting these results into X(z) leads to the final solution,

$$X(z) = \frac{z}{z-1} - \frac{z}{z-0.5}.$$

2. Shifting Property

Shifting a sequence (delaying or advancing) multiplies the z-transform by a power of z. If x(n)has a z-transform X (z),

$$x(n-n_0) \overset{Z}{\longleftrightarrow} z^{-n_0} X(z)$$

The z-transforms of x(n) and x (n - no) have the same region of convergence, except for adding or deleting the points z = 0 and z = ∞

Example 4: Determine the z-transform of the following sequence:

$$y(n) = (0.5)^{(n-5)} \cdot u(n-5),$$

where u(n- 5) = 1 for n ≥ 5 and u(n - 5) = 0 for n < 5.

Solution:

first use the shift theorem to have

$$Y(z) = Z\left[(0.5)^{n-5} u(n-5)\right] = z^{-5} Z[(0.5)^n u(n)].$$

using Table leads to:

$$Y(z) = z^{-5} \cdot \frac{z}{z-0.5} = \frac{z^{-4}}{z-0.5}.$$

3. Convolution

The two sequences x1(n) and x2(n), their convolution can be determined as follows:

$$x(n) = x_1(n) * x_2(n) = \sum_{k=0}^{\infty} x_1(n-k)x_2(k),$$

In z-transform domain, we have

$$X(z) = X_1(z)X_2(z).$$

Here, $X(z) = Z(x(n))$, $X_1(z) = Z(x_1(n))$, and $X_2(z) = Z(x_2(n))$.

Example 5: Given two sequences

$$x_1(n) = 3\delta(n) + 2\delta(n-1)$$
$$x_2(n) = 2\delta(n) - \delta(n-1),$$

Find the z-transform of their convolution

Solution:

Applying z-transform to $x_1(n)$ and $x_2(n)$, respectively, it follows that

$$X_1(z) = 3 + 2z^{-1}$$
$$X_2(z) = 2 - z^{-1}.$$

Using the convolution property, we have

$$X(z) = X_1(z)X_2(z) = (3 + 2z^{-1})(2 - z^{-1})$$
$$= 6 + z^{-1} - 2z^{-2}.$$

1.4. The Inverse z-Transform

The z-transform is a useful tool in linear systems analysis. However, just as important as techniques for finding the z-transform of a sequence are methods that may be used to invert the z-transform and recover the sequence x(n) from X(z).

$$X(z) = Z(x(n))$$
$$\text{and } x(n) = Z^{-1}(X(z)),$$

Where,
Z( ) is the z-transform operator,
And Z^{-1}( ) is the inverse z- transform operator.

The inverse z-transform may be obtained by at least three methods:

1. Partial fraction expansion and look-up table.
2. Power series expansion.

3. Residue method.

The general procedure of partial fraction expansion and look-up table is as follows:

1. Eliminate the negative powers of z for the z-transform function X(z).
2. Determine the rational function X(z)/z (assuming it is proper) and apply the partial fraction expansion to the determined rational function X(z)/z using the formula in the following Table.
3. Multiply the expanded function X(z)/z by z on both sides of the equation to obtain X(z).
4. Apply the inverse z-transform using Table 1.1.

Example 6: Find the inverse of the following z-transform

$$X(z) = \frac{1}{(1 - z^{-1})(1 - 0.5z^{-1})}.$$

Solution:

Eliminating the negative power of z by multiplying the numerator and denominator by z^2 yields

$$X(z) = \frac{z^2}{z^2(1 - z^{-1})(1 - 0.5z^{-1})} = \frac{z^2}{(z - 1)(z - 0.5)}$$

Dividing both sides by z leads to

zahraa.eisa@uomus.edu.iq

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)}.$$

Again, we write

$$\frac{X(z)}{z} = \frac{A}{(z-1)} + \frac{B}{(z-0.5)}.$$

Thus, to find A and B from this formula:

$$A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)}\bigg|_{z=1} = 2,$$

$$B = (z-0.5)\frac{X(z)}{z}\bigg|_{z=0.5} = \frac{z}{(z-1)}\bigg|_{z=0.5} = -1.$$

Thus

$$\frac{X(z)}{z} = \frac{2}{(z-1)} + \frac{-1}{(z-0.5)}.$$

Multiplying z on both sides gives

$$X(z) = \frac{2z}{(z-1)} + \frac{-z}{(z-0.5)}.$$

Using Table 5.1 of the z-transform pairs, it follows that

$$x(n) = 2u(n) - (0.5)^n u(n).$$

Example 7:Find *y(n)* if

$$Y(z) = \frac{z^2(z+1)}{(z-1)(z^2-z+0.5)}.$$

Solution:

Dividing *Y(z)* by z, we have

$$\frac{Y(z)}{z} = \frac{z(z+1)}{(z-1)(z^2-z+0.5)}.$$

Applying the partial fraction expansion leads to

$$\frac{Y(z)}{z} = \frac{B}{z-1} + \frac{A}{(z-0.5-j0.5)} + \frac{A^*}{(z-0.5+j0.5)}.$$

We first find B:

$$B = (z-1)\frac{Y(z)}{z}\bigg|_{z=1} = \frac{z(z+1)}{(z^2-z+0.5)}\bigg|_{z=1} = \frac{1\times(1+1)}{(1^2-1+0.5)} = 4.$$

Notice that A and A*form a complex conjugate pair. We determine A as follows:

$$A = (z-0.5-j0.5)\frac{Y(z)}{z}\bigg|_{z=0.5+j0.5} = \frac{z(z+1)}{(z-1)(z-0.5+j0.5)}\bigg|_{z=0.5+j0.5}$$

$$A = \frac{(0.5+j0.5)(0.5+j0.5+1)}{(0.5+j0.5-1)(0.5+j0.5-0.5+j0.5)} = \frac{(0.5+j0.5)(1.5+j0.5)}{(-0.5+j0.5)j}.$$

Using the polar form, we get

$$A = \frac{(0.707\angle 45^\circ)(1.58114\angle 18.43^\circ)}{(0.707\angle 135^\circ)(1\angle 90^\circ)} = 1.58114\angle -161.57^\circ$$
$$A^* = \bar{A} = 1.58114\angle 161.57^\circ.$$

Assume that a first-order complex pole has the form We have

$$P = 0.5 + 0.5j = |P|\angle\theta = 0.707\angle 45^\circ \text{ and } P^* = |P|\angle -\theta = 0.707\angle -45^\circ.$$

We have

$$Y(z) = \frac{4z}{z-1} + \frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)}.$$

Applying the inverse z-transform from line 15 in Table 1.1 leads to

$$y(n) = 4Z^{-1}\left(\frac{z}{z-1}\right) + Z^{-1}\left(\frac{Az}{(z-P)} + \frac{A^*z}{(z-P^*)}\right).$$

Using the previous formula, the inversion and subsequent simplification yield.

$$\begin{aligned} y(n) &= 4u(n) + 2|A|(|P|)^n \cos(n\theta + \phi)u(n) \\ &= 4u(n) + 3.1623(0.7071)^n \cos(45^\circ n - 161.57^\circ)u(n) \end{aligned}$$

zahraa.eisa@uomus.edu.iq