Al-Mustaqbal University / College of Engineering & Technology
Class: first
Subject: Differential Mathematics/Code: UOMU024013
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Lecture name: Derivatives
Lecture: 5
1st term

Applications of derivatives

1. Indeterminate Forms and L'Hopital's Rule

It is a rule in which derivatives are used to calculate limits of fractions whose numerators and denominators both approach zero or ∞. The rule is known today as l'Hôpital's Rule; i.e. l'Hôpital's Rule can be applied only to limits that give indeterminate forms.

Indeterminate Form 0/0

If the continuous functions $f(x)$ and $g(x)$ are both zero at $x = a$, then

$$\lim_{x \to a} \frac{f(x)}{g(x)}$$

cannot be found by substituting $x = a$. The substitution produces $0/0$, a meaningless expression, which we cannot evaluate. We use $0/0$ as a notation for an expression known as an **indeterminate form**. Sometimes, but not always, limits that lead to indeterminate forms may be found by cancellation, rearrangement of terms, or other algebraic manipulations. This was our experience in Chapter 2. It took considerable analysis in Section 2.4 to find $\lim_{x \to 0} (\sin x)/x$. But we have had success with the limit

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a},$$

from which we calculate derivatives and which always produces the equivalent of $0/0$ when we substitute $x = a$. L'Hôpital's Rule enables us to draw on our success with derivatives to evaluate limits that otherwise lead to indeterminate forms.

THEOREM 6 L'Hôpital's Rule (First Form)

Suppose that $f(a) = g(a) = 0$, that $f'(a)$ and $g'(a)$ exist, and that $g'(a) \neq 0$. Then

$$\lim_{x \to a} \frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}.$$

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Caution
To apply l'Hôpital's Rule to f/g, divide the derivative of f by the derivative of g. Do not fall into the trap of taking the derivative of f/g. The quotient to use is f'/g', not $(f/g)'$.

EXAMPLE The following limits involve $0/0$ indeterminate forms, so we apply l'Hôpital's Rule. In some cases, it must be applied repeatedly.

(a) $\lim_{x\to 0} \frac{3x - \sin x}{x} = \lim_{x\to 0} \frac{3 - \cos x}{1} = \frac{3 - \cos x}{1}\Big|_{x=0} = 2$

(b) $\lim_{x\to 0} \frac{\sqrt{1+x} - 1}{x} = \lim_{x\to 0} \frac{\frac{1}{2\sqrt{1+x}}}{1} = \frac{1}{2}$

(c) $\lim_{x\to 0} \frac{\sqrt{1+x} - 1 - x/2}{x^2}$ $\quad \frac{0}{0}$; apply l'Hôpital's Rule.

$= \lim_{x\to 0} \frac{(1/2)(1+x)^{-1/2} - 1/2}{2x}$ $\quad$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$= \lim_{x\to 0} \frac{-(1/4)(1+x)^{-3/2}}{2} = -\frac{1}{8}$ $\quad$ Not $\frac{0}{0}$; limit is found.

(d) $\lim_{x\to 0} \frac{x - \sin x}{x^3}$ $\quad \frac{0}{0}$; apply l'Hôpital's Rule.

$= \lim_{x\to 0} \frac{1 - \cos x}{3x^2}$ $\quad$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$= \lim_{x\to 0} \frac{\sin x}{6x}$ $\quad$ Still $\frac{0}{0}$; apply l'Hôpital's Rule again.

$= \lim_{x\to 0} \frac{\cos x}{6} = \frac{1}{6}$ $\quad$ Not $\frac{0}{0}$; limit is found.

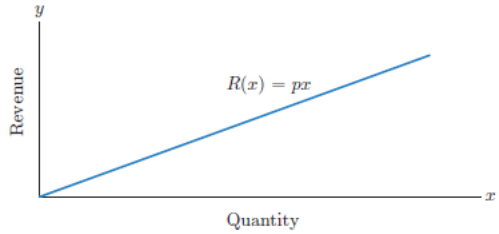
Ex.: Find the following limit $\lim_{x\to 0} 1 - \frac{\cos x}{x+x^2}$

Sol.: $\lim_{x\to 0} \frac{1-\cos x}{x+x^2} = \frac{0}{0}$

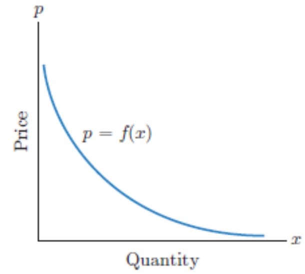
$=\lim_{x\to 0} \frac{\sin x}{1+2x} = \frac{0}{1+0} = 0$

2. Revenue Functions

If x units of a product are sold at a price p per unit, the total revenue $R(x)$ is given by: $R(x)=p.x$



A revenue curve.



A demand curve.

EXAMPLE Maximizing Revenue The demand equation for a certain product is $p = 6 - \frac{1}{2}x$ dollars. Find the level of production that results in maximum revenue.

SOLUTION In this case, the revenue function $R(x)$ is

$$R(x) = x \cdot p = x\left(6 - \frac{1}{2}x\right) = 6x - \frac{1}{2}x^2$$

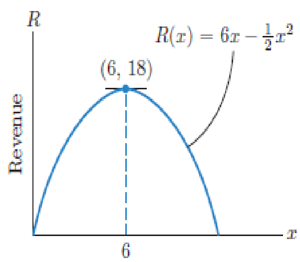
dollars. The marginal revenue is given by

$$R'(x) = 6 - x.$$

The graph of $R(x)$ is a parabola that opens downward. (See Fig. 6.) It has a horizontal tangent precisely at those x for which $R'(x) = 0$—that is, for those x at which marginal revenue is 0. The only such x is $x = 6$. The corresponding value of revenue is

$$R(6) = 6 \cdot 6 - \frac{1}{2}(6)^2 = 18 \text{ dollars.}$$

Thus, the rate of production resulting in maximum revenue is $x = 6$, which results in total revenue of 18 dollars.



Maximizing revenue.

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