Example8:

Find *x(n)* if

$$X(z) = \frac{z^2}{(z-1)(z-0.5)^2}.$$

Solution:

Dividing both sides of the previous z-transform by z yields

$$\frac{X(z)}{z} = \frac{z}{(z-1)(z-0.5)^2} = \frac{A}{z-1} + \frac{B}{z-0.5} + \frac{C}{(z-0.5)^2},$$

$$\text{where } A = (z-1)\frac{X(z)}{z}\bigg|_{z=1} = \frac{z}{(z-0.5)^2}\bigg|_{z=1} = 4.$$

$$B = R_2 = \frac{1}{(2-1)!}\frac{d}{dz}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

$$= \frac{d}{dz}\left(\frac{z}{z-1}\right)\bigg|_{z=0.5} = \frac{-1}{(z-1)^2}\bigg|_{z=0.5} = -4$$

$$C = R_1 = \frac{1}{(1-1)!}\frac{d^0}{dz^0}\left\{(z-0.5)^2\frac{X(z)}{z}\right\}_{z=0.5}$$

$$= \frac{z}{z-1}\bigg|_{z=0.5} = -1.$$

$$\text{Then } X(z) = \frac{4z}{z-1} + \frac{-4z}{z-0.5} + \frac{-1z}{(z-0.5)^2}.$$

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$$Z^{-1}\left\{\frac{z}{z-1}\right\} = u(n),$$

$$Z^{-1}\left\{\frac{z}{z-0.5}\right\} = (0.5)^n u(n),$$

$$Z^{-1}\left\{\frac{z}{(z-0.5)^2}\right\} = 2n(0.5)^n u(n).$$

From these results, it follows that

$$x(n) = 4u(n) - 4(0.5)^n u(n) - 2n(0.5)^n u(n).$$

1.4. Solution of Difference Equations Using the z-Transform

To solve a difference equation with initial conditions, we have to deal with time shifted sequences such as y(n - 1), y(n - 2), . . . , y(n - m), and so on. Let us examine the z-transform of these terms. Using the definition of the z-transform, we have

$$\begin{aligned} Z(y(n-1)) &= \sum_{n=0}^{\infty} y(n-1)z^{-n} \\ &= y(-1) + y(0)z^{-1} + y(1)z^{-2} + \ldots \\ &= y(-1) + z^{-1}\left(y(0) + y(1)z^{-1} + y(2)z^{-2} + \ldots\right) \end{aligned}$$

It holds that

$$Z(y(n-1)) = y(-1) + z^{-1}Y(z).$$

Similarly, we can have

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$$
\begin{aligned}
Z(y(n-2)) &= \sum_{n=0}^{\infty} y(n-2)z^{-n} \\
&= y(-2) + y(-1)z^{-1} + y(0)z^{-2} + y(1)z^{-3} + \ldots \\
&= y(-2) + y(-1)z^{-1} + z^{-2}\left(y(0) + y(1)z^{-1} + y(2)z^{-2} + \ldots\right) \\
Z(y(n-2)) &= y(-2) + y(-1)z^{-1} + z^{-2}Y(z) \\
Z(y(n-m)) &= y(-m) + y(-m+1)z^{-1} + \ldots + y(-1)z^{-(m-1)} \\
&\quad + z^{-m}Y(z),
\end{aligned}
$$

where y(- m), y(- m + 1), . . . , y(- 1) are the initial conditions. If all initial conditions are considered to be zero, that is,

$$y(-m) = y(-m+1) = \ldots y(-1) = 0,$$

Then

$$Z(y(n-m)) = z^{-m}Y(z),$$

The following two examples serve as illustrations of applying the z-transform to find the solutions of the difference equations. The procedure is:

1. Apply z-transform to the difference equation.
2. Substitute the initial conditions.
3. Solve for the difference equation in z-transform domain.
4. Find the solution in time domain by applying the inverse z-transform.

Example9:

A digital signal processing (DSP) system is described by the difference equation

$$y(n) - 0.5y(n-1) = 5(0.2)^n u(n).$$

Determine the solution when the initial condition is given by y(-1) = 1.

Solution:

Applying the z-transform on both sides of the difference equation

$$Y(z) - 0.5\left(y(-1) + z^{-1}Y(z)\right) = 5Z(0.2^n u(n)).$$

Substituting the initial condition and $Z(0.2^n u(n)) = z/(z - 0.2)$, we achieve

$$Y(z) - 0.5\left(1 + z^{-1}Y(z)\right) = 5z/(z-0.2).$$

Simplification yields

$$Y(z) - 0.5z^{-1}Y(z) = 0.5 + 5z/(z-0.2).$$

Factoring out Y(z) and combining the right-hand side of the equation, it follows that

$$Y(z) = \frac{(5.5z - 0.1)}{(1-0.5z^{-1})(z-0.2)} = \frac{z(5.5z-0.1)}{(z-0.5)(z-0.2)}.$$

Then we obtain

$$Y(z) = \frac{(5.5z - 0.1)}{(1-0.5z^{-1})(z-0.2)} = \frac{z(5.5z-0.1)}{(z-0.5)(z-0.2)}.$$

Using the partial fraction expansion method leads to

$$\frac{Y(z)}{z} = \frac{5.5z - 0.1}{(z-0.5)(z-0.2)} = \frac{A}{z-0.5} + \frac{B}{z-0.2},$$

$$A = (z-0.5)\frac{Y(z)}{z}\bigg|_{z=0.5} = \frac{5.5z-0.1}{z-0.2}\bigg|_{z=0.5} = \frac{5.5\times 0.5 - 0.1}{0.5-0.2} = 8.8333,$$

$$B = (z-0.2)\frac{Y(z)}{z}\bigg|_{z=0.2} = \frac{5.5z-0.1}{z-0.5}\bigg|_{z=0.2} = \frac{5.5\times 0.2 - 0.1}{0.2-0.5} = -3.3333.$$

Thus

$$Y(z) = \frac{8.8333z}{(z-0.5)} + \frac{-3.3333z}{(z-0.2)},$$

which gives the solution as

$$y(n) = 8.3333(0.5)^n u(n) - 3.3333(0.2)^n u(n).$$

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