Al-Mustaqbal University

College of Engineering & Technology

Biomedical Engineering Department

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Lecture Title: [SQUENCE]

Sequences

In mathematics, a **sequence** is a list of objects (or events) which have been ordered in a sequential fashion; such that each member either comes before, or after, every other member.

<u>*Example*</u>

$0, \quad 1, \quad 2, \quad . \quad . \quad . \quad n-1$ ➡ $a_1, a_2, a_3, a_4 \dots \dots . a_n$ **and** $a_n = n^{th}\ term = the\ term\ with\ index\ n$

$First\ term\ a_1 = 0$, $Second\ term\ a_2 = 1$, $Third\ term\ a_3 = 2$, $a_n = n - 1$, $n = 1,2,3,4, \dots \dots$

(**n** represents the domain (always positive integer), and **a_n** is the range)

<u>*Example*</u>

$1, \quad \frac{1}{2}, \quad \frac{1}{3}, \quad . \quad . \quad . \quad \frac{1}{n}$

$First\ term\ a_1 = 1$, $Second\ term\ a_2 = \frac{1}{2}$, $Third\ term\ a_3 = \frac{1}{3}$, $a_n = \frac{1}{n}$, $n = 1, 2, 3, 4, \dots \dots$

Examples

Example $a_n = \dfrac{n+1}{n}$ then the terms are

1^{st} *term* 2^{nd} *term* 3^{rd} *term* n^{th} *term*

$$a_1 = 2, \quad a_2 = \frac{3}{2}, \quad a_3 = \frac{4}{3}, \quad . \quad . \quad . \quad a_n = \frac{n+1}{n}, \quad . \quad . \quad .$$

and we use the notation $\{a_n\}$ as the sequence a_n .

Example

Find the first five terms of the following:

(a) $\left\{\dfrac{2n-1}{3n+2}\right\}$, (b) $\left\{\dfrac{1-(-1)^n}{n^3}\right\}$, (c) $\left\{(-1)^{n+1}\dfrac{x^{2n-1}}{(2n-1)!}\right\}$

Solution

(a) $\dfrac{1}{5}, \quad \dfrac{3}{8}, \quad \dfrac{5}{11}, \quad \dfrac{7}{14}, \quad \dfrac{9}{17}$

(b) $2, \quad 0, \quad \dfrac{2}{27}, \quad 0, \quad \dfrac{2}{125}$

(c) $x, \quad \dfrac{-x^3}{3!}, \quad \dfrac{x^5}{5!}, \quad \dfrac{-x^7}{7!}, \quad \dfrac{x^9}{9!}$

Examples

Example

Find the n^{th} $term\ \{a_n\}\ of\ the\ following\ sequence.$

12, 14, 16, 18, 20, ……

$a_n = 2n + 6$ at $n = 3, 4, 5, 6, 7$ … …. .

Or $a_n = 2n$ at $n = 6, 7, 8, 9, 10$ … . .

Example

Find the n^{th} $term\ \{a_n\}\ of\ the\ following\ sequences.$

(a) $0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4},$ (b) $0, \frac{\ln 2}{2}, \frac{\ln 3}{3}, \frac{\ln 4}{4},$ (c) $0, \frac{1}{4}, \frac{2}{9}, \frac{3}{16},$

(d) $2, 1, \frac{2^3}{3^2}, \frac{2^4}{4^2}, \frac{2^5}{5^2}$

Solution

(a) $a_n = \frac{n-1}{n},$ **(b)** $a_n = \frac{\ln n}{n},$ **(c)** $a_n = \frac{n-1}{n^2},$ **(d)** $a_n = \frac{2^n}{n^2}$ } All at $n = 1, 2, 3, 4,$ … ….

Convergence of Sequences

The fact that $\{a_n\}$ converges to L is written as

$$\lim_{n\to\infty} a_n = L \qquad \text{or} \qquad a_n \to L \ \text{ as } \ n \to \infty$$

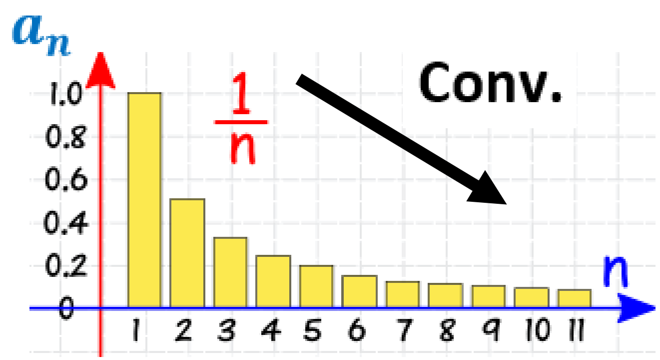
and we call the limit of the sequence $\{a_n\}$. If no such limit exists, we say that $\{a_n\}$ diverges.

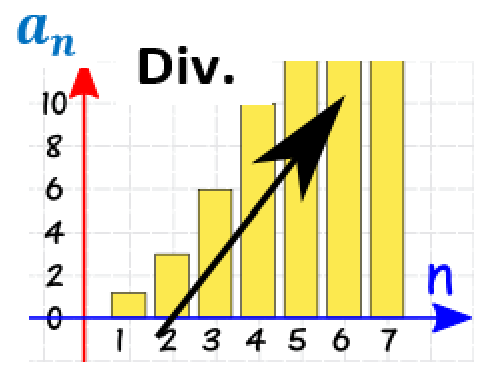
From that we can say that

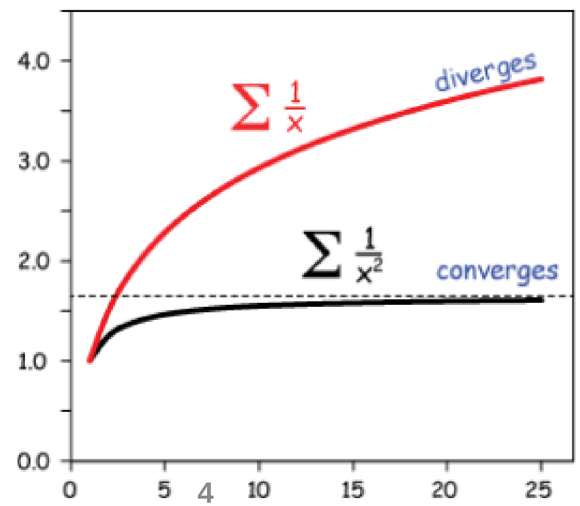
1) $\lim_{n\to\infty} a_n = \pm \mathrm{L}$ (Conv.)

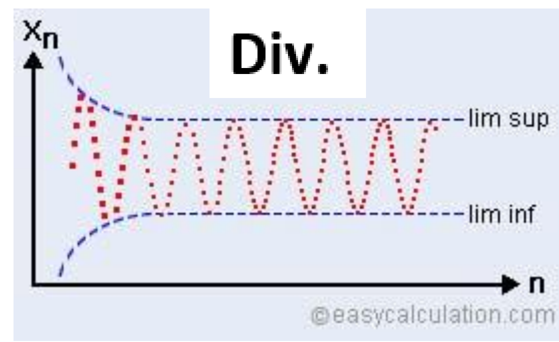
2) $\lim_{n\to\infty} a_n = \pm \infty$ (Div.)

3) $\lim_{n\to\infty} a_n = \begin{cases} L_1 \\ L_2 \end{cases}$ (Div.)









Convergence of Sequences

Also, if $A = \lim_{n\to\infty} a_n$ and $B = \lim_{n\to\infty} b_n$ both exist and are finite, then

i) $\lim_{n\to\infty}\{a_n + b_n\} = A + B$

ii) $\lim_{n\to\infty}\{ka_n\} = kA$

iii) $\lim_{n\to\infty}\{a_n \cdot b_n\} = A \cdot B$

iv) $\lim_{n\to\infty}\left\{\frac{a_n}{b_n}\right\} = \frac{A}{B}$, provided $B \neq 0$ and b_n is never 0

Note:- *The forms* $\left(\frac{0}{0}, \frac{\infty}{\infty}, \infty.0, 0^0, \infty - \infty, 1^\infty\right)$ *are meaningless expressions and called* ***indeterminate forms***.

If $\lim_{n\to\infty} a_n = \frac{0}{0}, \frac{\infty}{\infty}, \infty.0, 0^0, \infty - \infty, 1^\infty$, then the limit will **not exist**

and L'Hôpital's Rule should be used.

Examples

Example

Test the convergence of the following:

(a) $\left\{\frac{1}{n}\right\}$, (b) $\left\{1+(-1)^n\right\}$, (c) $\left\{n^2\right\}$, (d) $\left\{\sqrt{n+1}-\sqrt{n}\right\}$,

(e) $\left\{\frac{3n^2-5n}{5n^2+2n+6}\right\}$, (f) $\left\{\frac{3n^2-4n}{2n-1}\right\}$, (g) $\left\{\left(\frac{2n-3}{3n-7}\right)^4\right\}$, (h) $\left\{\frac{2n^5-4n^2}{3n^7+n^2-10}\right\}$,

(i) $\left\{\frac{2^n}{5n}\right\}$, (j) $\left\{\frac{\ln n}{e^n}\right\}$

Examples

1)	$\lim_{n\to\infty} a_n = L$	(Conv.)
2)	$\lim_{n\to\infty} a_n = \infty$	(Div.)
3)	$\lim_{n\to\infty} a_n = \begin{cases} L_1 \\ L_2 \end{cases}$	(Div.)

Solution

(a) $\lim_{n\to\infty} \underbrace{\left(\frac{1}{n}\right)}_{a_n} = 0$ ***(Conv.)***

(b) $\lim_{n\to\infty} \underbrace{\left(1 + (-1)^n\right)}_{a_n} = 1 + \lim_{n\to\infty} (-1)^n = \begin{cases} 0 & n \ \ odd \\ 2 & n \ \ even \end{cases}$ ***(Div.)***

(c) $\lim_{n\to\infty} \underbrace{\left(n^2\right)}_{a_n} = \infty$ ***(Div.)***

(d) $\lim_{n\to\infty} \underbrace{\left(\sqrt{n+1} - \sqrt{n}\right)}_{a_n} = \lim_{n\to\infty} \left(\left(\sqrt{n+1} - \sqrt{n}\right) \times \frac{\sqrt{n+1} + \sqrt{n}}{\sqrt{n+1} + \sqrt{n}} \right) = \lim_{n\to\infty} \left(\frac{n+1-n}{\sqrt{n+1} + \sqrt{n}} \right)$

$$= \lim_{n\to\infty} \left(\frac{1}{\sqrt{n+1} + \sqrt{n}} \right) = \frac{1}{\infty + \infty} = 0$$ ***(Conv.)***

Examples

1) $\lim_{n\to\infty} a_n = L$ (Conv.)

2) $\lim_{n\to\infty} a_n = \infty$ (Div.)

3) $\lim_{n\to\infty} a_n = \begin{cases} L_1 \\ L_2 \end{cases}$ (Div.)

(e) $\lim_{n\to\infty}\underbrace{\left(\frac{3n^2-5n}{5n^2+2n+6}\right)}_{a_n} = \lim_{n\to\infty}\left(\frac{\frac{3n^2}{n^2}-\frac{5n}{n^2}}{\frac{5n^2}{n^2}+\frac{2n}{n^2}+\frac{6}{n^2}}\right) = \frac{3}{5}$ ***(Conv.)***

(f) $\lim_{n\to\infty}\underbrace{\left(\frac{3n^2-4n}{2n-1}\right)}_{a_n} = \lim_{n\to\infty}\left(\frac{\frac{3n^2}{n^2}-\frac{4n}{n^2}}{\frac{2n}{n^2}-\frac{1}{n^2}}\right) = \frac{3}{0} = \infty$ ***(Div.)***

(g) $\lim_{n\to\infty}\underbrace{\left(\frac{2n-3}{3n-7}\right)^4}_{a_n} = \left(\frac{2}{3}\right)^4 = \frac{16}{81}$ ***(Conv.)***

We can also apply L'Hôpital's Rule

Examples

1) $\lim_{n\to\infty} a_n = L$ (Conv.)

2) $\lim_{n\to\infty} a_n = \infty$ (Div.)

3) $\lim_{n\to\infty} a_n = \begin{cases} L_1 \\ L_2 \end{cases}$ (Div.)

(h) $\lim_{n\to\infty}\left(\frac{2n^5-4n^2}{3n^7+n^2-10}\right) = \lim_{n\to\infty}\left(\frac{\frac{2}{n^2}-\frac{4}{n^5}}{3+\frac{1}{n^5}-\frac{10}{n^7}}\right) = 0$ ***(Conv.)***

We can also apply L'Hôpital's Rule

(i) $\lim_{n\to\infty}\left(\frac{2^n}{5n}\right) = \lim_{n\to\infty}\left(\frac{2^n.\ln 2}{5}\right) = \infty$ ***(Div.)***

(j) $\lim_{n\to\infty}\left(\frac{\ln n}{e^n}\right) = \lim_{n\to\infty}\left(\frac{1/n}{e^n}\right) = \lim_{n\to\infty}\left(\frac{1}{n.e^n}\right) = \frac{1}{\infty} = 0$ ***(Conv.)***

L'Hôpital's Rule is applied

H.W.

Find the values of a_1, a_2, a_3 and a_4 for the following sequences

1) $a_n = \frac{1-n}{n^2}$

2) $a_n = \frac{1}{n!}$

3) $a_n = \frac{(-1)^{n+1}}{2n-1}$

4) $a_n = 2 + (-1)^n$

5) $a_n = \frac{2^n}{2^{n+1}}$

6) $a_n = \frac{2^n - 1}{2^n}$

Find the n^{th} *term* $\{a_n\}$ *of the following sequences.*

1) $1, -1, 1, -1, 1, \ldots.$

2) $-1, 1, -1, 1, -1, \ldots.$

3) $1, -4, 9, -16, 25, \ldots$

4) $1, -\frac{1}{4}, \frac{1}{9}, -\frac{1}{16}, \frac{1}{25}, \ldots$

5) $0, 3, 8, 15, 24, \ldots$

6) $-3, -2, -1, 0, 1, \ldots$

H.W.

Which of the following sequences converge and which diverge?

1) $a_n = 2 + (0.1)^n$

2) $a_n = \dfrac{1-2n}{1+2n}$

3) $a_n = \dfrac{1-5n^4}{n^4+8n^3}$

4) $a_n = \dfrac{n^2-2n+1}{n-1}$

5) $a_n = 1 + (-1)^n$

6) $a_n = \left(\dfrac{n+1}{2n}\right)\left(1-\dfrac{1}{n}\right)$

7) $a_n = \dfrac{(-1)^{n+1}}{2n-1}$

8) $a_n = \sqrt{\dfrac{2n}{n+1}}$

Increasing, Decreasing, and Bounded Sequences

In the previous lesson we studied the sequences and we understood the sequences limits, the convergence, and the divergence. In this lesson we will study the increasing, decreasing, and bounded sequences.

Def 1 : A sequence $\{a_n\}$ is called an increasing sequence (non-decreasing) if $a_n \leq a_{n+1}$ $\forall n$. In another word $a_1 \leq a_2 \leq a_3 \leq \cdots$.

Def 2 : A sequence $\{a_n\}$ is called decreasing sequence (non-increasing) if $a_n \geq a_{n+1}$ $\forall n$. In another word $a_1 \geq a_2 \geq a_3 \geq \cdots$.

Def 3 : If $\{a_n\}$ an increasing or decreasing sequence then is called monotonic sequence.

Def 4 : A sequence $\{a_n\}$ is bounded from above if $\exists$ $M \in R$ s.t. $a_n \leq M$ $\forall n$, the number M is an upper bounded for $\{a_n\}$. If M is upper bounded for $\{a_n\}$, but no number less than M is an upper bounded for $\{a_n\}$, then M is the least upper bounded for $\{a_n\}$.

Def 5 : A sequence $\{a_n\}$ is bounded from below if $\exists$ $m \in R$ s.t. $a_n \geq m$ $\forall n$, the number m is lower bounded for $\{a_n\}$. If m is lower bounded for $\{a_n\}$, but no number greater than m is a lower bounded for $\{a_n\}$, then m is the greatest lower bounded for $\{a_n\}$.

Def 6 : If $\{a_n\}$ is bounded from above and below, then $\{a_n\}$ is bounded. If $\{a_n\}$ is not bounded, then we say that $\{a_n\}$ is an unbounded sequence.

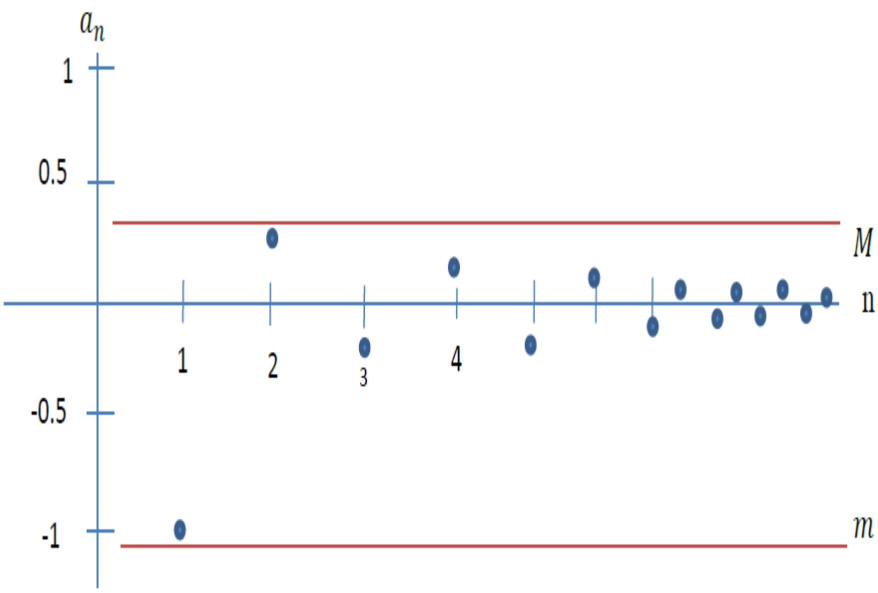
Note: The sequence to be increasing it must be increasing for all n. This means the all terms have to be increasing. Also, the sequence is monotonic, if it is increasing only or decreasing only.

Ex: Show that which of the following sequences is monotonic and bounded.

1. $\left\{(-1)^n \frac{1}{n^2}\right\}_{n=1}^{\infty}$

Sol.

$$\left\{(-1)^n \frac{1}{n^2}\right\}_{n=1}^{\infty} \rightarrow \{-1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{16}, -\frac{1}{25}, \dots\}.$$



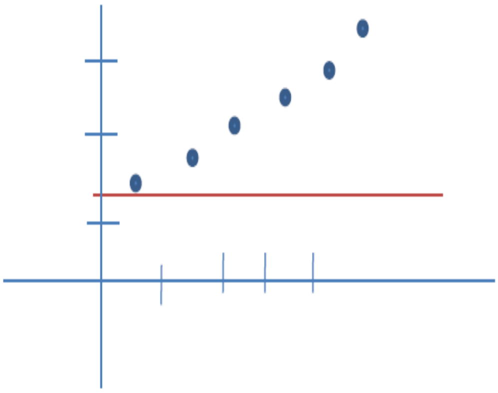
It is neither increasing nor decreasing, so it is not a monotonic sequence.

It is bounded above at $\{\frac{1}{4}\}$, and bounded below at $\{-1\}$. So the sequence is bounded.

We want to test the sequence if it is converges or diverges.

2. $\left\{\frac{3n}{n+2}\right\}_{n=1}^{\infty}$

Sol. $\left\{\frac{3n}{n+2}\right\}_{n=1}^{\infty} \rightarrow \{1, 1.5, 1.8, 2, \dots\}$



$a_1 = \frac{3(1)}{1+2} = \frac{3}{3} = 1$

$a_2 = \frac{3(2)}{2+2} = \frac{6}{4} = \frac{3}{2} = 1.5$

$a_3 = \frac{3(3)}{3+2} = \frac{9}{5} = 1.8$

$a_4 = \frac{3(4)}{4+2} = \frac{12}{6} = 2$

The sequence is bounded below at 1, but it's not bounded above.

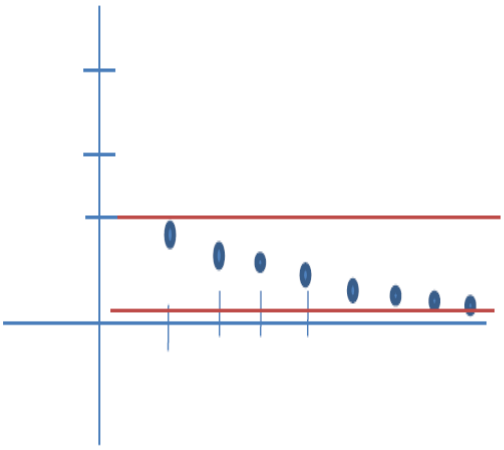
The sequence is increasing so it is monotonic.

Another way to test the increasing

3. $\left\{\left(\frac{3}{4}\right)^{n}\right\}_{n=1}^{\infty}$

Sol. $\left\{\left(\frac{3}{4}\right)^{n}\right\}_{n=1}^{\infty} \rightarrow \left\{\frac{3}{4}, \frac{9}{16}, \frac{27}{64}, \frac{81}{256}, \cdots\right\}$

$\rightarrow \{0.75, 0.6, 0.4, 0.3, \dots\}$



As it shows that the sequence is decreasing

But let test it

$a_n \geq a_{n+1}$.

$\frac{a_n}{a_{n+1}} \geq 1$